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Fractional Calculus in Different Fields: Applications

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Abstract. While integer-order calculus represents the established theoretical framework for describing dynamic processes across engineering, physics, biomedicine and finance, it lacks the expressive capacity to deliver accurate mathematical representations of systems whose evolution is governed by pronounced memory effects and non-local spatial interactions. The idea of fractional calculus consists in the generalization of the derivatives and integrals to non-integer orders, thus providing tools for modeling systems with memory and non-local action. The present article will review the applications of fractional calculus in four areas of research: viscoelasticity and fractional-order control, anomalous diffusion, pharmacokinetics and drug delivery, and long-range dependence in time series of financial assets. The approach used in this paper will be a bibliographic survey of experiments and empirical studies comparing fractional models to classical ones. The fractional constitutive relations describe a wider range of frequencies and strain rates compared to Voigt and Kelvin models. Time and space fractional diffusion equations account for the anomalous scaling, which cannot be provided by the Fick's law. Mittag-Leffler compartmental models describe the power-law drug elimination kinetics and multiple half-lives of the compound.

Keywords: fractional calculus, memory effect, anomalous diffusion, pharmacokinetics, long-range dependence

1. Introduction

Calculus is a fundamental mathematical concept that enables one to study change and accumulation, having found wide applications in various physical and natural phenomena. Unfortunately, many phenomena in the real world are characterized by their complex temporal or spatial behavior, which cannot be accurately described using standard integer-order approaches. Fractional calculus extends classical calculus by allowing derivatives and integrals to take non-integer orders, providing greater flexibility for mathematical modeling. Although the foundations of this branch of mathematics are more than a few centuries old, fractional calculus has evolved from being purely theoretical to becoming a modeling framework for complex systems [1, 2].

An important property of fractional calculus is that it captures memory and non-locality effects. While most classical theories describe a system by considering only its current status, fractional operators can take into account the system's past status or even more general regions of the system. Therefore, it makes fractional calculus an appropriate tool for dealing with systems that are affected by their history or spatial interaction [2, 3].

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This paper discusses the use of fractional calculus in various fields. Engineering, physics, biology and medicine, as well as finance and economics are some of the areas where the use of fractional calculus is discussed. This literature review analyzes some applications and to find out the mathematical features that enable fractional calculus to be used in these fields.

2. Fundamental characteristics of fractional calculus

2.1. Fractional derivatives and integrals

The fractional calculus includes a set of fractional derivatives and integrals such as the Riemann-Liouville, Caputo, and Grünwald-Letnikov operators. Each operator differs from another one based on its definition and properties. Thus, one can choose any formulation based on the nature of the problem to be modeled [4]. Hence, one may choose any formulation depending on the nature of the problem to be modeled [4]. Instead of employing one specific operator, fractional calculus offers a mathematical approach capable of describing various processes.

2.2. Memory and non-locality

One distinguishing feature of fractional derivatives is that they are non-local in nature. Unlike integer-order derivatives that are normally based on local data, fractional derivatives may be formulated with the help of data obtained throughout an interval instead of a point [5]. The use of such non-local operators in the time domain may serve as a memory effect such that the present state of a system depends on its past states. This is especially true for systems with long-term dependence.

2.3. Modeling flexibility

The flexibility of fractional calculus arises due to its fractional order, non-locality, and memory properties. There is freedom to choose different types of fractional operators and orders in order to model various types of temporal or spatial dependence [6]. This allows fractional calculus to be used for modeling those systems where conventional integer-order models would not be adequate.

3. Applications of fractional calculus in different fields

3.1. Engineering — viscoelasticity and control systems

Deformation exhibited by viscoelastic solids applied in engineering constructions is influenced by the past loading history rather than the stress instantaneously applied. Traditional integer-order models like the Voigt or Kelvin model establish the relation between stress and strain using ordinary derivatives and hence are capable of describing only a restricted set of relaxation phenomena. In general, these models are not able to predict the rate dependence of relaxation behavior of polymers and soft solids over a wide range of frequency or strain-rate.

Fractional constitutive equations use fractional derivatives as opposed to integer-order derivatives in establishing the relation between stress and strain. These operators have a memory effect, which implies that the current value of stress is influenced by past strain history. This approach has been validated against experimental data. The study carried out on tissue mimicking materials like CF-11 and gelatin using surface wave techniques proved that fractional viscoelastic models accurately capture wave dynamics better than Voigt and Kelvin models over the range of frequencies studied [7]. A comparison for the dielectric elastomer VHB 4910 reached a similar conclusion: a fractional

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model remained accurate across deformation rates spanning four orders of magnitude, whereas the companion integer-order model lost accuracy outside a narrower rate interval [8]. However, there is an important caveat to include in the impulse experiments on the polymer gel. A number of fractional equations retrieve the acceleration signal obtained at the initial stage, but their projections diverge at the peak of the response for small compression rates, the right operator will then be dependent on the medium and the loading condition [9].

The same is true about control systems design. Fractional Proportional-Integral-Derivative(PID) PID controllers involve non-integer orders of the integral and derivative parts, thus providing for a larger family of possible feedbacks compared to the traditional PID system [10]. Here, the fractional order becomes a new adjustable parameter bridging two limiting integer-order effects.

3.2. Physics — anomalous diffusion

Fick's law and the Gaussian relation for the mean squared displacement and time characterize the standard diffusion. For a heterogeneous medium, however, the above relation is commonly substituted by a power-law relationship $\langle x^2(t) \rangle \propto t^{\alpha}$, $\alpha \neq 1$, and the process is called anomalous diffusion: $0< \alpha <1$ is subdiffusion, $\alpha >1$ is superdiffusion. Since the classical diffusion equation is local both in space and time, it is incapable of producing the scaling behavior.

Analogously, fractional kinetic equations describe the above phenomena in a continuum framework. From the continuous time random walks, Metzler and Klafter obtained the fractional diffusion and Fokker–Planck equations where the order of the time derivative reflects the long-range waiting time, whereas the order of the spatial differential operator captures the long-range jumps [11]. Thus, the time-fractional derivative describes the memory in the transport process, while the space-fractional derivative describes the non-locality of the transport process.

These experimental observations confirm the validity of the proposed formulation. In a one-dimensional optical lattice, the spread of an ultracold atomic sample showed a power-law increase in time, and the spatial shape displayed fractional self-similarity that corresponds to heavy tails of the probability distribution function [12]. Methanol diffusion into a packed zeolite catalyst did not obey Fickian dynamics either, it could be modeled by a diffusion equation with a time-fractional derivative or with time- and space-fractional derivatives with the time order being about 0.85 and corresponding to subdiffusion [13]. The order estimation based on tracer data obtained in a porous medium at Columbus Air Force Base revealed the same trend, the deviation of the model from data was smallest for a time order of about 0.6 and a space order of about 2 [14]. The last two examples impose a constraint. Purely space-fractional equations did not outperform Fick's law in the case of the catalyst pellets, therefore, the order of the operator cannot be selected arbitrarily but should correspond to the major source of nonlocality in the physical process.

3.3. Biology and medicine — pharmacokinetics and drug delivery

Classical compartmental pharmacokinetics models absorption, distribution, and elimination via first-order ordinary differential equations. Consequently, the concentration exhibits decay according to a combination of exponential functions and thus possesses a clearly defined half-life. This does not hold for several pharmaceutical compounds: terminal plasma profiles exhibit a power-law behaviour, the concentration lacks a half-life, and multiple doses may lead to irregular accumulation. Amiodarone is an established example of such anomalous pharmacokinetics.

The replacement of certain integer-order rates in compartmental models by Caputo fractional derivatives of order less than one leads to fractional compartmental models. A fractional first-order

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process solution is described by a Mittag-Leffler function, which shows stretched exponential behaviour in the short term and power-law behaviour in the long term [15]. Multi-compartmental fractional models are constructed analogously. In case the rate of return from a peripheral compartment is fractional, then the model describes deep-tissue sequestration of the pharmaceutical compound without additional empirical parameters, and fits to intravenous concentration-time data have been reported [16]. The practical implication is obvious: without classical half-life, the predicted accumulation in constant or multiple dose schedules is different. Consequently, the dosing regimen must be re-designed accordingly [17].

Pharmacokinetics of drug release from solid dosage forms encounters the same challenge. If the diffusion through the inhomogeneous matrix does not follow the Fickian regime, and the kinetic model is of fractional order, then the release process will exhibit Mittag–Leffler kinetics once again. The analysis of the Mittag-Leffler function in comparison with the Weibull function of fractal kinetics demonstrates the fact that both describe the experimental release profiles until around 85% of the initially loaded drug, afterwards, both models start to overestimate the amount of drug released [18].

3.4. Finance and economics — long-range dependence and financial modeling

The standard models for asset prices assume that logarithmic increments are independent. The geometric Brownian motion and Black–Scholes equation are local in time: the present return is independent of the distant past, and no information about waiting times between transactions is included in the model. However, the financial data violate both assumptions. Correlations of either absolute or squared returns decay slowly, and the waiting times between successive transactions have a power-law distribution instead of the exponential one.

Two fractional models accommodate these properties. The fractional Brownian motion substitutes the ordinary Brownian motion by a Gaussian process with correlated increments if the Hurst exponent is not equal to one half, exponents greater than one half are associated with long-range dependence [19]. Another approach involves the construction of the continuous-time random walk for tick-by-tick prices. Under the diffusion approximation, the probability distribution of the log-price obeys a fractional kinetic equation which is non-Markovian in time and non-local in space, and hence belongs to the same family as the models of anomalous diffusion discussed in Section 3.2 [20]. Waiting-time density functions, predicted by the equation, have been matched with the actual waiting time intervals between trades of BUND futures at LIFFE and found to agree [21]. Semiparametric estimates using a panel data set of 34 equity indices show that fractional behavior is present in a significant number of these markets [22]. Applying fractional behavior in an option price model and examining it with respect to Shanghai 50 ETF options results in less pricing error when using a nonparametric fractional approach than the classical Black-Scholes model [23].

The application in finance faces the same constraint that was encountered in the earlier applications. Fractional Brownian motion with Hurst parameter value different from one-half is not a semimartingale, and so, the replicating procedure of the classical theory cannot apply directly. Long-memory behavior has also been shown more strongly to be a feature of volatility and waiting time intervals than raw signed returns. The fractional order must thus be estimated using the specific market series, it is not a universal phenomenon in all financial markets.

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4. Why fractional calculus works across different fields

The four applications mentioned in Section 3 include polymers, tracers, drugs and prices. Yet the demands of modelling remain the same. One needs memory, the interaction goes beyond the neighbourhood, and there should be an additional real parameter to interpolate between classical cases.

4.1. Memory effects

The fractional derivative with respect to time is an integral of previous states. It is used to model history dependent stress in viscoelastic solids [7], rare large waiting times in anomalous diffusion and in trading pauses [11, 21], deep tissue retention in anomalous diffusion [16] and slow decay in autocorrelation of absolute returns. Integer-order rates take into account the current state only, so they lack the memory that is available in all these models.

4.2. Non-local interaction

If the same integral is defined with respect to space rather than time, it couples far away states. A space fractional diffusion is a model of long jumps of particles, the tick-by-tick continuous time random walk of Section 3.4 is the corresponding finance application [20]. Memory corresponds to the non-locality in time, while jumps correspond to the non-locality in space.

4.3. Fractional order and modelling versatility

The order is the modelling parameter. It bridges a spring with a dashpot, Fickian and anomalous scaling, exponential and power-law distribution, and Brownian and long memory prices. The extra orders of a fractional PID controller fulfill the same purpose in feedback synthesis [10]. Altering the order changes the regime while keeping the form of the equation the same.

4.4. Mathematical similarities among various scientific disciplines

In all four domains the data need a causal convolution kernel with slowly decaying memory. The fractional calculus gives such kernel in one shot, by another name: a constitutive operator, a fractional Fokker-Planck equation, a Mittag-Leffler compartment, or a fractional diffusion limit for log-prices. The interdisciplinary scope arises out of the universal kernel, and not out of any physical similarity of the applications.

5. Conclusion

The current literature review summarized the application of fractional calculus in engineering, physics, biology and medicine, and finance, and pointed out the properties of mathematics, which make it possible to use the same operators in all mentioned domains. The above comparison is based on published experiments and fits, rather than on any newly conducted experiments.

In engineering, fractional viscoelasticity modeling of the movement of surface waves in the materials of CF-11 and gelatin, and the response of the elastomer VHB 4910 at different rates of strain over four decades were better described by using fractional models compared to integer-order Voigt and Kelvin laws. However, fractional equations gave distinct results at the peak of the impulse response test of polymer gel, so the appropriate operator depends on the material and loading. A

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fractional-order PID (PI^λD^μ) controller incorporates two non-integer order parameters, which significantly expand the feasible design space for feedback control loop shaping . The anomalous scaling behavior of the mean squared displacement in fractional diffusion processes, alongside Fokker-Planck equations derived from continuous-time random walks, has been well-established in the physics literature . Multiple experimental systems, including ultracold atoms in optical lattices , methanol loading in zeolite pellets, and tracer dispersion in porous media , can be rigorously described by time-fractional differential equations. In contrast, pure spatial fractionalization of Fick's law fails to deliver improved descriptive performance for catalyst pellet systems. In pharmacokinetics, power-law decay due to Caputo derivatives of orders less than one led to the Mittag–Leffler kinetics for a number of drugs, including amiodarone. It was the Mittag–Leffler kinetics that described drug release from a solid dosage form until approximately 85% of the loaded amount but failed for the remaining fraction along with the Weibull kinetics, where the latter one slightly underestimated the residual amount. In financial modeling, both fractional Brownian motion and the fractional diffusion limit derived from tick-by-tick continuous-time random walks are capable of capturing the slowly decaying autocorrelations and heavy-tailed waiting times widely observed in empirical market data. Empirical evidence from Shanghai 50 ETF option markets demonstrates that a nonparametric fractional pricing model yields significantly lower pricing errors than the classical Black-Scholes framework. However, it is well-documented that any Hurst exponent deviating from 0.5 breaks the semimartingale property of the underlying price process. These four distinct application scenarios do not share a unified underlying physical principle in their respective implementations. Nevertheless, all of them are subject to an identical core modeling constraint: the adoption of a causal convolution kernel characterized by power-law memory decay, alongside a non-integer operation order that lies strictly between the two classical integer-order limiting cases. Further deployment of this modeling framework relies on two core procedures: calibrating the fractional order parameter to the specific empirical dataset, and validating the appropriateness of the selected operator using late-time or high-frequency observation data, rather than a priori assuming the inherent superiority of non-integer-order operators.

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