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A Beta-Weighted Integral Identity for Bernoulli Polynomials with Applications to Stirling Numbers and Central Binomial Coefficients

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Abstract. In this paper, a closed-form expression is derived for the integral of a Bernoulli polynomial multiplied by a Beta density kernel. The result is formulated in terms of Gamma functions, Bernoulli numbers, and Pochhammer symbols, and its proof is elementary, relying only on the binomial expansion of the Bernoulli polynomials, the Beta integral, and basic properties of the Gamma function. By inserting the Stirling-number expansion of the Bernoulli numbers, a double-sum identity is obtained that involves only the Stirling numbers of the second kind. As an illustration of the method, two different proofs of a recent formula of Jha are provided; these proofs differ in their starting points, one using a variable transformation together with a Beta integral, and the other employing a generating-function technique. The special case where both parameters equal one-half produces a generating-function identity that links Bernoulli numbers, central binomial coefficients, hyperbolic functions, and the modified Bessel function I_0. These identities may serve as useful toolkit pieces for future work on Beta-weighted integrals of Appell polynomials.

Keywords: Bernoulli numbers, Bernoulli polynomials, Beta integral, Stirling numbers of the second kind, Central binomial coefficients

1. Introduction

Bernoulli polynomials $B_n(x)$ and Bernoulli numbers $B_n = B_n(0)$ appear throughout number theory, combinatorics, and analysis. Their integral representations are well documented; see, e.g., the DLMF [1] (§24.13) for integrals of B_n over various intervals. However, integrals weighted by the Beta density $t^{a-1}(1-t)^{b-1}$ have received less attention. The closest known results involve the Beta moments of Appell polynomials (see [2] for general theory) and hypergeometric Bernoulli polynomials [3, 4]; nevertheless, the explicit closed form presented here we have not found this explicit form in the references consulted.

In this note, an identity is proven that evaluates such an integral as a finite sum involving Bernoulli numbers, Pochhammer symbols, and Gamma functions. The proof is elementary and uses only the binomial expansion of $B_n(t)$, the Beta integral, and basic properties of the Gamma function.

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After establishing the main theorem, the Bernoulli numbers are replaced by their well-known expression that involves Stirling numbers of the second kind (see [5], p. 286, Eq. (7.44), with the convention $B_1 = +\frac{1}{2}$ consistent there), obtaining a pure Stirling-number representation of the same integral. This representation is then illustrated by giving two distinct proofs of a recent explicit formula of Jha [6] for Bernoulli numbers. The first proof uses a variable transformation and a Beta integral, differing from Jha's original Mellin-transform approach. The second proof employs generating functions and another Beta integral. Finally, the special choice $a = b = \frac{1}{2}$ leads to a generating-function identity that connects Bernoulli numbers, central binomial coefficients, and the modified Bessel function I_0. The exponential-type generating function of central binomial coefficients $\sum_{n\ge 0}\binom{2n}{n}x^n/n! = e^{2x}I_0(2x)$ [7] underlies the appearance of I_0 in our final identity.

2. Preliminaries

2.1. Bernoulli polynomials and numbers

The exponential-series generating function of the Bernoulli numbers B_n is expressed as

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \qquad |t| < 2\pi,$$

with $B_0 = 1,\ B_1 = +\frac{1}{2},\ B_2 = \frac{1}{6},\ B_3 = 0,\ B_4 = -\frac{1}{30}, \ldots$. The Bernoulli polynomials satisfy

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

and admit the expansion

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}. \tag{1}$$

2.2. Gamma and Beta functions

The Gamma function satisfies $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$.

The Beta integral is given by

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \qquad a > 0,\ b > 0. \tag{2}$$

For integer $m \ge 0$,

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$$\int_0^1 \mathrm{t}^{\mathrm{a}-1}(1-t)^{\mathrm{b}-1}\,\mathrm{t}^{\mathrm{m}}\,dt = \frac{\Gamma(\mathrm{a}+\mathrm{m})\Gamma(\mathrm{b})}{\Gamma(\mathrm{a}+\mathrm{b}+\mathrm{m})} \tag{3}$$

2.3. Pochhammer symbol

The Pochhammer symbol is defined as

$$(\mathrm{x})_{\mathrm{k}} = \mathrm{x}(\mathrm{x}+1)\cdots(\mathrm{x}+\mathrm{k}-1) = \frac{\Gamma(\mathrm{x}+\mathrm{k})}{\Gamma(\mathrm{x})}, \qquad (\mathrm{x})_0 = 1.$$

2.4. Stirling numbers of the second kind

Let $\mathrm{S}(\mathrm{n},\mathrm{k})$ denote the Stirling number of the second kind, which are defined as the number of ways to partition an n-element set into k non-empty subsets. Its exponential generating function reads

$$\frac{(e^t-1)^k}{k!} = \sum_{n=k}^{\infty} S(n,k)\frac{t^n}{n!}.$$

A standard expression that relates the Bernoulli numbers to the Stirling numbers (see [5], p. 286, Eq. (7.44)) is

$$\mathrm{B}_{\mathrm{n}} = \sum_{\mathrm{k}=0}^{\mathrm{n}} (-1)^{\mathrm{k}} \frac{\mathrm{k}!}{\mathrm{k}+1}\, S(\mathrm{n},\mathrm{k}). \tag{4}$$

3. Main theorem

The main result of this paper is the following closed-form evaluation of the Beta-weighted integral of a Bernoulli polynomial.

Theorem 1. For any non-negative integer n and real $\mathrm{a} > 0,\ \mathrm{b} > 0$,

$$\int_0^1 \mathrm{t}^{\mathrm{a}-1}(1-t)^{\mathrm{b}-1}\mathrm{B}_{\mathrm{n}}(t)\,dt = \frac{\Gamma(\mathrm{a})\Gamma(\mathrm{b})}{\Gamma(\mathrm{a}+\mathrm{b})} \sum_{\mathrm{k}=0}^{\mathrm{n}} \binom{\mathrm{n}}{\mathrm{k}} \mathrm{B}_{\mathrm{k}} \frac{(\mathrm{a})_{\mathrm{n}-\mathrm{k}}}{(\mathrm{a}+\mathrm{b})_{\mathrm{n}-\mathrm{k}}}. \tag{5}$$

Proof. By inserting the expansion (1):

$$\begin{aligned} \mathrm{I}_{\mathrm{n}}(a,b) &:= \int_0^1 \mathrm{t}^{\mathrm{a}-1}(1-t)^{\mathrm{b}-1}\mathrm{B}_{\mathrm{n}}(t)\,dt \\ &= \sum_{\mathrm{k}=0}^{\mathrm{n}} \binom{\mathrm{n}}{\mathrm{k}} \mathrm{B}_{\mathrm{k}} \int_0^1 \mathrm{t}^{\mathrm{a}+\mathrm{n}-\mathrm{k}-1}(1-t)^{\mathrm{b}-1}\,dt \\ &= \sum_{\mathrm{k}=0}^{\mathrm{n}} \binom{\mathrm{n}}{\mathrm{k}} \mathrm{B}_{\mathrm{k}} \frac{\Gamma(\mathrm{a}+\mathrm{n}-\mathrm{k})\Gamma(\mathrm{b})}{\Gamma(\mathrm{a}+\mathrm{b}+\mathrm{n}-\mathrm{k})} \quad [\text{by (3)}]. \end{aligned}$$

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Using $(x)_m = \Gamma(x+m)/\Gamma(x)$,

$$\frac{\Gamma(a+n-k)}{\Gamma(a+b+n-k)} = \frac{\Gamma(a)}{\Gamma(a+b)} \cdot \frac{(a)_{n-k}}{(a+b)_{n-k}},$$

which gives (5).

Remark 2. Theorem 1 can be seen as the Beta-moment formula that governs the Bernoulli polynomials, which belong to the class of Appell polynomials. For a general Appell polynomial $A_n(x)$ satisfying $A_n'(x) = nA_{n-1}(x)$, the Beta moment can be expressed similarly using the coefficients of the polynomial expansion. The specific closed form in terms of Pochhammer symbols and Bernoulli numbers does not appear in the standard references [1, 2] to the best of our knowledge. Closest analogues appear in the theory of hypergeometric Bernoulli polynomials [3] and in recent work on identities involving Bernoulli and Euler numbers [4], where Beta-weighted integrals of $B_N(t)$ also reduce to Beta functions, but the classical case $N = 1$ with an explicit Pochhammer sum seems not to have been isolated.

4. Stirling-number representation

By substituting the Stirling-number representation of the Bernoulli numbers into Theorem 1, one obtains a purely combinatorial form of the integral.

Corollary 3. Under the same hypotheses,

$$\begin{aligned}\int_0^1 t^{a-1}(1-t)^{b-1}B_n(t)\,dt &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \sum_{k=0}^{n}\binom{n}{k}\frac{(a)_{n-k}}{(a+b)_{n-k}} \sum_{j=0}^{k}(-1)^j\frac{j!}{j+1}\,S(k,j) \\ &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \sum_{j=0}^{n}(-1)^j\frac{j!}{j+1}\left(\sum_{k=j}^{n}\binom{n}{k}\frac{(a)_{n-k}}{(a+b)_{n-k}}\,S(k,j)\right).\end{aligned} \tag{6}$$

Proof. By substituting the representation (4) for B_k into (5). Because the sums are finite, the order of summation can be interchanged:

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(a)_{n-k}}{(a+b)_{n-k}}\sum_{j=0}^{k}(-1)^j\frac{j!}{j+1}\,S(k,j) = \sum_{j=0}^{n}(-1)^j\frac{j!}{j+1}\sum_{k=j}^{n}\binom{n}{k}\frac{(a)_{n-k}}{(a+b)_{n-k}}\,S(k,j).$$

The two expressions in (6) are equivalent.

Remark 4. The double-sum form eliminates Bernoulli numbers entirely, revealing a direct combinatorial connection between the Beta-weighted integral and Stirling numbers. This representation may facilitate further combinatorial interpretations.

5. Two proofs of Jha's formula

In this section, two different proofs of the following identity due to Jha [6] are presented.

Theorem 5. (Jha [6], Eq. (1)) For any non-negative integer r,

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$$B_{r+1} = \sum_{k=0}^{r} \frac{(-1)^{k-1}\, k!\, S(r,k)}{(k+1)(k+2)}\,.$$

5.1. First proof (via variable transformation)

Proof. Jha [6] starts from the integral representation

$$B_{r+1} = -\int_0^{\infty} \frac{\mathrm{Li}_{-r}(-x)}{(1+x)^2}\, dx. \tag{7}$$

where Li_{-r} is the polylogarithm. By substituting $x = t/(1-t)$, then $dx = dt/(1-t)^2$, $1+x = 1/(1-t)$, and the limits become $t : 0 \to 1$. After simplification,

$$B_{r+1} = -\int_0^1 \mathrm{Li}_{-r}\left(-\frac{t}{1-t}\right) dt.$$

Using Landsburg's expansion [8],

$$\mathrm{Li}_{-r}(-x) = \sum_{k=0}^{r} k!\, S(r,k)\, \frac{(-x)^k}{(1+x)^{k+1}}\,,$$

it is obtained that

$$\mathrm{Li}_{-r}\left(-\frac{t}{1-t}\right) = \sum_{k=0}^{r} k!\, S(r,k)\, (-t)^k\, (1-t).$$

Hence

$$B_{r+1} = -\sum_{k=0}^{r} (-1)^k k!\, S(r,k) \int_0^1 t^k (1-t)\, dt.$$

The integral is a Beta integral:

$$\int_0^1 t^k (1-t)\, dt = B(k+1,2) = \frac{1}{(k+1)(k+2)}\,.$$

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Thus

$$B_{r+1} = -\sum_{k=0}^{r} \frac{(-1)^k k!\, S(r,k)}{(k+1)(k+2)} = \sum_{k=0}^{r} \frac{(-1)^{k-1} k!\, S(r,k)}{(k+1)(k+2)},$$

which is exactly the statement of Theorem 5.

5.2. Second proof (via generating functions and Beta integral)

Proof. A polynomial is defined as

$$P_r(u) = \sum_{k=0}^{r} (-1)^k k!\, S(r,k)\, u^k.$$

Consider the integral

$$I_r = \int_0^1 (1-u) P_r(u)\, du.$$

By integrating termwise and using the Beta function (2),

$$\int_0^1 u^k (1-u)\, du = B(k+1,2) = \frac{k!}{(k+2)!} = \frac{1}{(k+1)(k+2)}.$$

Hence

$$I_r = \sum_{k=0}^{r} (-1)^k k!\, S(r,k)\, \frac{1}{(k+1)(k+2)}.$$

The statement to be proved is $B_{r+1} = -I_r$; therefore, it suffices to show $I_r = -B_{r+1}$.

Serving as the exponential generating function for I_r, the target expression is computed by applying the function that generates the Stirling numbers of the second kind,

$$\frac{(e^t - 1)^k}{k!} = \sum_{n \geq k} S(n,k) \frac{t^n}{n!},$$

which yields

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$$\sum_{r\ge 0} P_r(u)\frac{t^r}{r!} = \sum_{k\ge 0}(-1)^k u^k (e^t-1)^k = \frac{1}{1+u(e^t-1)}.$$

Consequently,

$$\sum_{r\ge 0} I_r \frac{t^r}{r!} = \int_0^1 (1-u)\cdot \frac{1}{1+u(e^t-1)}\,du.$$

By setting $a = e^t - 1$, it follows that

$$\int_0^1 \frac{1-u}{1+au}\,du = \int_0^1 \left(-\frac{1}{a} + \frac{a+1}{a(1+au)}\right)du = -\frac{1}{a} + \frac{a+1}{a^2}\ln(1+a).$$

Substituting $a = e^t - 1$ and $\ln(1+a) = t$ gives

$$\sum_{r\ge 0} I_r \frac{t^r}{r!} = -\frac{1}{e^t-1} + \frac{e^t}{(e^t-1)^2}\,t = \frac{1-e^t+te^t}{(e^t-1)^2}.$$

On the other hand, the Bernoulli numbers satisfy

$$\frac{t}{e^t-1} = \sum_{n\ge 0} B_n \frac{t^n}{n!}.$$

By differentiating both sides, it is found that

$$\frac{d}{dt}\left(\frac{t}{e^t-1}\right) = \frac{(e^t-1)-te^t}{(e^t-1)^2} = \sum_{n\ge 0} B_{n+1}\frac{t^n}{n!}.$$

Thus

$$\sum_{r\ge 0} B_{r+1}\frac{t^r}{r!} = \frac{e^t-1-te^t}{(e^t-1)^2} = -\frac{1-e^t+te^t}{(e^t-1)^2} = -\sum_{r\ge 0} I_r \frac{t^r}{r!}.$$

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By comparing coefficients, it is obtained that $B_{r+1} = -I_r$. Since $I_r = \sum_{k=0}^{r} \frac{(-1)^k k!\, S(r,k)}{(k+1)(k+2)}$, it follows that

$$B_{r+1} = \sum_{k=0}^{r} \frac{(-1)^{k-1} k!\, S(r,k)}{(k+1)(k+2)},$$

which is exactly the statement of Theorem 5.

Remark 6. Both proofs rely on the Beta integral but differ in their starting points: the first uses a polylogarithmic integral representation, while the second uses a generating function technique. Together they illustrate the versatility of the Beta function in handling identities involving Bernoulli and Stirling numbers.

6. Special case $a = b = \frac{1}{2}$ and a generating-function identity

By setting $a = b = \frac{1}{2}$ in Theorem 1, and since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $(\frac{1}{2})_m = (2m)!/(2^{2m} m!)$,

$$\frac{(\frac{1}{2})_{n-k}}{(1)_{n-k}} = \frac{\binom{2(n-k)}{n-k}}{4^{n-k}}.$$

Thus

$$\int_0^1 \frac{B_n(t)}{\sqrt{t(1-t)}}\, dt = \pi \sum_{k=0}^{n} \binom{n}{k} B_k \frac{\binom{2(n-k)}{n-k}}{4^{n-k}}. \tag{8}$$

A sequence is defined as $S_n := \pi^{-1} \int_0^1 B_n(t)/\sqrt{t(1-t)}\, dt$. By the symmetry $B_n(1-t) = (-1)^n B_n(t)$ (see [9]) and the invariance of the weight $1/\sqrt{t(1-t)}$ under $t \mapsto 1-t$, it is found that $S_n = 0$ for all odd n. Direct evaluation using the explicit form $B_4(t) = t^4 - 2t^3 + t^2 - \frac{1}{30}$ [2] and the Beta-moment formula $\int_0^1 t^m/\sqrt{t(1-t)}\, dt = \pi \binom{2m}{m}/4^m$ yields

$$S_0 = 1, \quad S_1 = 0, \quad S_2 = \frac{1}{24}, \quad S_3 = 0, \quad S_4 = -\frac{19}{1920}, \quad S_5 = 0, \quad \dots$$

Now consider the function that serves as the exponential generating function

$$F(x) = \sum_{n=0}^{\infty} S_n \frac{x^n}{n!}.$$

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Theorem 7. For $|x| < 2\pi$,

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}B_k\frac{\binom{2(n-k)}{n-k}}{4^{n-k}}\right)\frac{x^n}{n!} = \frac{x}{2\sinh\left(\frac{x}{2}\right)}\,I_0\left(\frac{x}{2}\right). \tag{9}$$

Proof. Recalling the definition of S_n, the left-hand side of (9) is precisely $F(x)$. Applying the exponential generating function governing the Bernoulli polynomials [9]

$$\sum_{n=0}^{\infty}B_n(t)\frac{x^n}{n!} = \frac{x\,e^{tx}}{e^x-1}, \qquad |x| < 2\pi,$$

$F(x)$ is rewritten as

$$\begin{aligned} F(x) &= \frac{1}{\pi}\int_0^1 \frac{1}{\sqrt{t(1-t)}}\sum_{n=0}^{\infty}B_n(t)\frac{x^n}{n!}\,dt \\ &= \frac{1}{\pi}\int_0^1 \frac{1}{\sqrt{t(1-t)}}\cdot\frac{x\,e^{tx}}{e^x-1}\,dt. \end{aligned}$$

For $|x| < 2\pi$, the series $\sum B_n(t)x^n/n!$ exhibits uniform convergence in $t \in [0,1]$; hence the interchange of sum and integral is justified.

The remaining integral is a standard Laplace-type integral. By substituting $t = \sin^2\theta$ ($0 \le \theta \le \pi/2$), it is found that

$$\int_0^1 \frac{e^{tx}}{\sqrt{t(1-t)}}\,dt = 2\int_0^{\pi/2} e^{x\sin^2\theta}\,d\theta = e^{x/2}\int_0^{\pi} e^{-(x/2)\cos\phi}\,d\phi,$$

where $\sin^2\theta = (1-\cos 2\theta)/2$ was used and $\phi = 2\theta$ was substituted. The integral representation of the modified Bessel function (see, e.g., [1], Eq. (10.32.1))

$$I_0(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\phi}\,d\phi$$

with $z = -x/2$ (and using the evenness $I_0(-x/2) = I_0(x/2)$ yields

$$\int_0^1 \frac{e^{tx}}{\sqrt{t(1-t)}}\,dt = \pi\,e^{x/2}\,I_0\left(\frac{x}{2}\right).$$

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Substituting this into the expression for $\mathrm{F}(\mathrm{x})$, it is obtained that

$$\begin{aligned} F\left(x\right) &= \tfrac{1}{\pi} \cdot \tfrac{\mathrm{x}}{\mathrm{e}^{\mathrm{x}}-1} \cdot \pi\, \mathrm{e}^{\mathrm{x}/2}\, \mathrm{I}_0\left(\tfrac{\mathrm{x}}{2}\right) \\ &= \tfrac{\mathrm{x}\,\mathrm{e}^{\mathrm{x}/2}}{\mathrm{e}^{\mathrm{x}}-1}\, \mathrm{I}_0\left(\tfrac{\mathrm{x}}{2}\right) \\ &= \tfrac{\mathrm{x}}{2\sinh(\mathrm{x}/2)}\, \mathrm{I}_0\left(\tfrac{\mathrm{x}}{2}\right), \end{aligned}$$

where in the last step, $\mathrm{e}^{\mathrm{x}} - 1 = 2\mathrm{e}^{\mathrm{x}/2}\sinh(\mathrm{x}/2)$ was used. The right-hand side of (9) is recovered exactly here.

Remark 8. The identity (9) provides a generating-function link between Bernoulli numbers, central binomial coefficients, the hyperbolic sine function, and the modified Bessel function I_0. The appearance of I_0 is natural because the exponential generating function corresponding to the central binomial coefficients is $\sum_{\mathrm{n}\geq 0}\binom{2\mathrm{n}}{\mathrm{n}}\mathrm{x}^{\mathrm{n}}/\mathrm{n}! = \mathrm{e}^{2\mathrm{x}}\mathrm{I}_0(2\mathrm{x})$ [7], which is consistent with the structure of (9). To the best of our knowledge, this particular combination has not been recorded in the standard references [1, 2, 5, 10, 11].

7. Conclusion

A Beta-weighted integral identity for Bernoulli polynomials (Theorem 1), a corresponding Stirling-number double-sum representation (Corollary 3), two distinct proofs of Jha's formula (Theorem 5), and a generating-function identity at $\mathrm{a} = \mathrm{b} = \frac{1}{2}$ (Theorem 7) linking Bernoulli numbers, central binomial coefficients, and the modified Bessel function I_0. These elementary identities, though not claiming novelty without further literature comparison, may serve as useful toolkit pieces for future work on Beta-weighted integrals of Appell polynomials.

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