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Group Representations in Two- and Three-Dimensional Kakeya Geometry

Shuxuan Wang

School of Mathematics and Statistics, University of Melbourne, Melbourne, Australia
shuxuanwang@student.unimelb.edu.au

Abstract. Kakeya geometry asks how small a subset of Euclidean space can be while containing a unit line segment in every direction. The problem is central to harmonic analysis because its tube formulations are closely related to maximal operators, Fourier restriction, and oscillatory integrals. This paper provides an analysis of the two-dimensional and three-dimensional Kakeya problems from the perspectives of group action and representation theory. This paper first passes from Kakeya sets to finite families of thin tubes, where dimension is studied through tube number, multiplicity, and union volume. It then describes an oriented line by its direction and transverse position and identify the line space as the homogeneous space $L_n^+ \cong SE(n)/(\mathbb{R} \times SO(n-1))$. The Euclidean action on points and lines induces quasi-regular representations, and the X-ray transform becomes an intertwining operator satisfying $X\pi(g) = \widehat{\pi}(g)X$. In two dimensions, this framework organizes translation invariance, Fourier analysis, and angular almost orthogonality. In three dimensions, spherical harmonics and affine rescaling organize directional and scale information, while additional positional and multiscale correlations explain why symmetry alone is insufficient. The resulting formulation does not provide a new Kakeya estimate; instead, it gives a unified language for comparing the planar and spatial problems and for isolating the geometric difficulties that remain in higher dimensions.

Keywords: Kakeya conjecture, Group representations, X-ray transform, Homogeneous spaces

1. Introduction

The Kakeya problem began with the question of how little area is needed to rotate a unit needle in the plane. Its modern set-theoretic form asks instead for the smallest possible dimension of a set containing a unit line segment in every direction. Although Besicovitch-type sets may have Lebesgue measure zero, the Kakeya conjecture predicts that they must have full Hausdorff and Minkowski dimension. The problem occupies a central position in harmonic analysis because estimates for thin tubes are closely connected with Fourier restriction, Bochner–Riesz summation, oscillatory integrals, local smoothing, and dispersive partial differential equations. The planar conjecture is classical, while the three-dimensional set conjecture was established only recently after a long sequence of geometric and analytic advances [1].

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Group representation theory provides a systematic language for converting geometric symmetries into linear operators. The Fourier transform diagonalizes the translation action, representations of rotation groups organize angular variables, and induced or quasi-regular representations describe analysis on homogeneous spaces. These ideas have been especially successful in integral geometry: Radon and X-ray transforms can be formulated between spaces on which the same group acts, and their covariance becomes an intertwining relation between representations [2, 3]. Since Kakeya geometry is built from translated and rotated line segments, this framework offers a natural way to compare its planar and three-dimensional forms without claiming that representation theory replaces the geometric estimates in their proofs.

This paper develops that viewpoint in two and three dimensions. Section 2 reviews the origin of the Kakeya problem, its set and maximal-function formulations, and the basic tube geometry that distinguishes the planar and spatial cases. Section 3 identifies oriented line spaces as homogeneous spaces of Euclidean motion groups and proves the equivariance of the X-ray transform. Section 3.1 explains the planar L^2 theory through translations, rotations, and angular almost orthogonality, while Section 3.2 discusses spherical directions, affine rescaling, and structured tube families in three dimensions. Section 4 summarizes the scope and limitations of this reformulation and suggests possible extensions to higher-dimensional homogeneous line spaces.

2. The Kakeya problem: origins, formulations, and basic methods

2.1. From the needle problem to Kakeya sets

The original problem asks for the smallest planar region in which a unit line segment can be rotated through 180°. Besicovitch-type constructions show that such directional line segments may be arranged inside sets of arbitrarily small areas. There are even compact sets of Lebesgue measure zero that contain a unit segment in every direction. Area is therefore too coarse to measure their geometric size [3].

This observation leads to the following definition.

Definition 2.1 (Kakeya set). A compact set $K \subset \mathbb{R}^n$ is called a Kakeya, or Besicovitch, set if, for every direction $\omega \in S^{n-1}$, there is a unit line segment

$$I_\omega = \{a_\omega + t\omega : 0 \le t \le 1\} \tag{1}$$

contained in K.

The point a_ω may depend arbitrarily on the direction. This freedom is the source of much difficulty: the line segments need not pass through a common point, and their positions may be chosen so that many of them overlap.

The modern Kakeya set conjecture replaces Lebesgue measure by fractal dimension.

Conjecture 2.2 (Kakeya set conjecture). Every Kakeya set $K \subset \mathbb{R}^n$ has full Hausdorff and Minkowski dimension:

$$\dim_{\mathrm{H}} K = \dim_{\mathrm{M}} K = n. \tag{2}$$

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Hausdorff dimension is defined through efficient coverings of K by sets of varying diameters. Minkowski dimension instead measures the rate at which the volume of the δ -neighbourhood

$$K_\delta = x \in \mathbb{R}^n : dist\,(x, K) < \delta \tag{3}$$

decays as $\delta \to 0$. If K behaves like a d -dimensional set, one heuristically expects $|K_\delta| \approx \delta^{n-d}$. Thus, full Minkowski dimension means that $|K_\delta|$ cannot decay like a fixed positive power of δ, even though $|K|$ itself may equal zero.

The conjecture is classical in the plane. Davies proved that every planar Kakeya set has Hausdorff dimension 2 [4]. The three-dimensional problem required substantially more geometric information. Wolff proved the lower bound $5/2$ [5], and Katz, Łaba, and Tao obtained the first improvement beyond this threshold [6]. Wang and Zahl later proved the sticky three-dimensional case and, in 2025, established that every Kakeya set in $\mathbb{R}^3$ has Hausdorff and Minkowski dimension 3 [7]. They then verified a sticky theorem one year later [8]. The corresponding conjecture remains open in general dimensions $n \geq 4$.

2.2. Discretization by thin tubes

Fix $0 < \delta \ll 1$, a direction $\omega \in S^{n-1}$, and a point $a \in \mathbb{R}^n$. Let

$$I_{\omega,a} = a + t\omega : 0 \leq t \leq 1, \tag{4}$$

and define the associated δ -tube by

$$T^\delta_{\omega,a} = x \in \mathbb{R}^n : dist\,(x, I_{\omega,a}) < \delta. \tag{5}$$

It has length comparable to 1, cross-sectional radius comparable to δ, and volume $\left|T^\delta_{\omega,a}\right| \asymp \delta^{n-1}$.

A maximal δ -separated collection of directions in S^{n-1} has approximately $\delta^{-(n-1)}$ elements. If one tube is chosen in each direction, then $\sum_{T\in\mathbb{T}_\delta}|T| \asymp 1$. Thus, the difficulty is not the total tube volume counted with multiplicity. It is whether many tubes can be placed so that they overlap and their union becomes much smaller.

Define the multiplicity function

$$m_{\mathbb{T}}(x) = \sum_{T\in\mathbb{T}_\delta} 1_T(x). \tag{6}$$

Its first moment satisfies $\int m_{\mathbb{T}} \asymp 1$, whereas its second moment records pairwise intersections:

$$\int m_{\mathbb{T}}(x)^2\,dx = \sum_{T,T'\in\mathbb{T}_\delta}|T \cap T'|. \tag{7}$$

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Cauchy–Schwarz gives $|\bigcup_{T\in\mathbb{T}_\delta} T| \gtrsim \frac{1}{\int m_{\mathbb{T}}^2}$. The dimension problem has therefore become a quantitative question about the number, overlap, and union volume of tubes. A standard finite-scale target is $|\bigcup_{T\in\mathbb{T}_\delta} T| \gtrsim_\varepsilon \delta^\varepsilon$ for every $\varepsilon > 0$.

If a Kakeya set is thickened by δ and a δ -separated family of its directional segments is selected, this estimate prevents any fixed power-law decay of the neighborhood volume and yields full Minkowski dimension.

The operator version is the Kakeya maximal function

$$M_\delta f(\omega) = \sup_{a\in\mathbb{R}^n} \frac{1}{|T^\delta_{\omega,a}|} \int_{T^\delta_{\omega,a}} |f(x)|\,dx. \tag{8}$$

The expected estimate is

$$\| M_\delta f \|_{L^n(S^{n-1})} \lesssim_\varepsilon \delta^{-\varepsilon} \| f \|_{L^n(\mathbb{R}^n)}. \tag{9}$$

This conjecture is stronger than the corresponding set statement. In particular, the solution of the three-dimensional set conjecture does not automatically settle the full three-dimensional maximal-function conjecture.

2.3. Direction, position, and symmetry

A tube is determined not only by its direction but also by its position. For a complete oriented line, it is useful to write

$$l_{\omega,p} = p + t\omega : t \in R, \qquad \omega \in S^{n-1}, \quad p \in \omega^\perp. \tag{10}$$

Here ω gives the direction, while p gives the point of the line nearest the origin. The direction has $n-1$ degrees of freedom, and the transverse position has another $n-1$. Hence the space of oriented lines has dimension $2n-2$.

This parameter count already suggests a difference between the plane and three-dimensional space. In $\mathbb{R}^2$, the direction and position each have one parameter. In $\mathbb{R}^3$, each has two. More importantly, rotations and translations act simultaneously on both variables. Rather than treating every line separately, one can organize all oriented lines as a single orbit of the Euclidean motion group. The next section makes this observation precise and lifts the geometric action to representations on function spaces.

3. Group representations in low-dimensional Kakeya geometry

The Euclidean motion group is $SE(n) = \mathbb{R}^n \rtimes SO(n)$. An element $g = (b, R)$ acts on points by $g \cdot x = Rx + b$. The same transformation acts on directions, segments, tubes, and oriented lines. This can be used to fix the reference line $l_0 = \mathbb{R}e_n$.

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Every oriented line is obtained from ℓ_0 by first rotating its direction and then translating its position. The subgroup preserving ℓ_0 consists of translations along the line and rotations fixing its direction. Thus

$$H_n \cong R \times SO(n-1), \tag{11}$$

and the oriented line space is the homogeneous space

$$\mathscr{L}_n^+ \cong SE(n)/(\mathbb{R} \times SO(n-1)). \tag{12}$$

This quotient description organizes all oriented lines into one group orbit. A function on $\mathscr{L}_n^+$ can equivalently be regarded as a function on $SE(n)$ that is constant on the right cosets of H_n. Geometric action therefore induces a quasi-regular representation on functions over line space [2].

More explicitly, define representations on physical space and line space by

$$(\pi(g)f)(x) = f(g^{-1}x), \qquad (\widehat{\pi}(g)F)(\ell) = F(g^{-1}\ell). \tag{13}$$

Euclidean motions preserve Lebesgue measure, line length, and the natural measure on oriented line space, so these are unitary on the corresponding L^2 spaces.

For a smooth, compactly supported function, define the X-ray transform by

$$(Xf)(\ell) = \int_\ell f(x)\, ds(x). \tag{14}$$

The transform is compatible with the group action. Indeed,

$$(X\pi(g)f)(l) = \int_\ell f(g^{-1}x)\, ds(x) = \int_{g^{-1}\ell} f(y)\, ds(y) = (\widehat{\pi}(g)Xf)(l). \tag{15}$$

Therefore

$$X\pi(g) = \widehat{\pi}(g)X. \tag{16}$$

The X-ray transform is an intertwining operator between the two quasi-regular representations. This is the central representation-theoretic identity of the paper. It states that integrating after moving a function is equivalent to moving the line data after integration. Helgason's theory of Radon transforms on homogeneous spaces and later work on dual pairs provides the broader setting for this construction, including boundedness and inversion questions [2, 3].

Intertwining is a symmetry statement, not by itself an estimate. It does not determine the norm of the transform, the size of tube intersections, or the effect of taking a supremum over tube positions.

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Those questions require additional analytic and geometric information. The value of the representation-theoretic formulation is that it separates the exact covariance from the dimension-dependent estimates.

3.1. The two-dimensional Kakeya geometry

In the plane, $SE(2) = \mathbb{R}^2 \rtimes SO(2), \mathscr{L}_2^+ \cong SE(2)/\mathbb{R}$. An oriented line can be parameterized by an angle θ and a signed distance s:

$$l_{\theta,s} = s\theta^{\perp} + t\theta : t \in R. \tag{17}$$

The X-ray transform becomes

$$(Xf)(\theta, s) = \int_{\mathbb{R}} f(s\theta^{\perp} + t\theta)\, dt. \tag{18}$$

Rotations change θ, while translations change s. The intertwining relation states that these changes can be performed either before or after integration. The translation subgroup is diagonalized by the Fourier transform. If $\tau_b f(x) = f(x - b)$, then

$$\widehat{\tau_b f}(\xi) = e^{-ib\cdot\xi}\widehat{f}(\xi). \tag{19}$$

Rotations act by moving the frequency variable: $\widehat{\pi(0,R)f}(\xi) = \widehat{f}(R^{-1}\xi)$. On S^1, the irreducible representations of $SO(2)$ are the characters $e^{ik\theta}$. Thus, Fourier analysis handles translation, while Fourier series on the circle organize angular variation.

This structure is compatible with the classical planar Kakeya argument. If two unit δ -tubes make an angle α, then

$$|T_1 \cap T_2| \lesssim \tfrac{\delta^2}{\delta+\alpha}\,. \tag{20}$$

For a fixed tube, there are $O(2^j)$ δ -separated directions at angle comparable to $2^j\delta$, and each intersection has area $O(\delta/2^j)$. Every dyadic angular scale therefore contributes $O(\delta)$, and summing the scales gives a logarithmic loss. In Fourier space, the same phenomenon appears as limited overlap between frequency regions associated with separate directions. Plancherel's theorem converts this separation into angular almost orthogonality, leading to Córdoba's estimate [9]

$$\| M_\delta f \|_{L^2(S^1)} \lesssim (log(1/\delta))^{1/2} \| f \|_{L^2(\mathbb{R}^2)}. \tag{21}$$

Tube averages themselves are equivariant. If

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$$A_{I,\delta}f = \frac{1}{|T_\delta(I)|}\int_{T_\delta(I)} f(x)\,dx, \tag{22}$$

then, for $g \in SE(2)$,

$$A_{gI,\delta}(\pi(g)f) = A_{I,\delta}f. \tag{23}$$

The Kakeya maximal function nevertheless takes a supremum over all positions of I with a fixed direction. This nonlinear step is not an intertwining operator. The group-action viewpoint therefore explains how translations, rotations, and line averages fit together, but the intersection estimate remains essential. Similarly, Davies's full Hausdorff-dimension result is compatible with the symmetry framework rather than a formal consequence of the representation theory of $SE(2)$ [4].

3.2. The three-dimensional Kakeya geometry

In three dimensions, $SE(3) = \mathbb{R}^3 \rtimes SO(3)$ *and* $\mathscr{L}_3^+ \cong SE(3)/(\mathbb{R} \times SO(2))$. The line space is four-dimensional: two parameters specify a direction in S^2, and two specify a transverse position. The direction sphere is itself the homogeneous space $S^2 \cong SO(3)/SO(2)$. The rotation representation on $L^2(S^2)$ decomposes into spherical harmonics $L^2(S^2) = \widehat{\bigoplus}_{\ell=0}^{\infty} \mathscr{H}_\ell$, where $\mathscr{H}_\ell$ is an irreducible $SO(3)$ -representation of dimension $2\ell + 1$. This is the three-dimensional analogue of decomposing a function on S^1 into Fourier modes. It separates low and high angular frequencies and is useful for rotation-invariant linear operators [2].

However, a spherical harmonic records variation with direction, not the transverse position of a tube. Tubes with separated directions may still cluster near a plane, a ruled surface, or a multiscale arrangement. The difference from the plane is therefore not merely the replacement of one angular parameter by two. Three-dimensional analysis must control direction, position, scale, and multiplicity simultaneously.

Affine transformations provide an additional useful action for scale changes. If $g(x) = Ax + b$, $A \in GL_3(\mathbb{R})$, then

$$(U_g f)(x) = |detA|^{-1/2} f(g^{-1}x). \tag{24}$$

defines a unitary representation on $L^2(\mathbb{R}^3)$. An anisotropic choice of A can normalize a thick parent tube and transform its thinner descendants to a relative scale. Directions change through the projective action $[\omega] \mapsto [A\omega]$.

If $F_{\mathbb{T}} = \sum_{T\in\mathbb{T}} 1_T$ is the overlap function, then $F_{g\mathbb{T}} = |\det A|^{1/2} U_g F_{\mathbb{T}}$. This exact covariance formula records how tube multiplicity transforms under an affine change of variables. A general affine image has an ellipsoidal cross-section, but an adapted rescaling makes it comparable with a standard tube at the new scale.

These actions explain why rotations, translations, and rescalings recur throughout three-dimensional Kakeya arguments, but they do not control how many tubes can concentrate near a common structure. Wolff's hairbrush method and the sticky theorem require geometric information

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beyond symmetry [5, 8]. The later Assouad-dimension result [7], the full three-dimensional proof adds incidence geometry [9], multiscale non-concentration, and projection estimates. None of these results follows merely by decomposing a representation of $SE(3)$ or $SO(3)$.

This can also be explored further from a geometric perspective. Projection theory supplies part of the additional geometry, such as radial projection estimates are developed in [10], and restricted projections indexed by curves in higher dimensions.

The comparison is now clear. In two dimensions, the one-dimensional direction space and the geometry of pairwise intersections fit naturally with an L^2 almost-orthogonality argument. In three dimensions, spherical representations still organize angular information, but the additional positional freedom permits coherent configurations involving many tubes and many scales [11]. Group representations provide a common language for both settings; the difference in difficulty lies in the geometric information that remains after symmetry has been identified.

4. Conclusion

This paper has developed a unified group-action and representation-theoretic interpretation of Kakeya geometry in two and three dimensions. This work began with the dimension problem for sets containing a unit segment in every direction, thickened the segments into δ -tubes, and converted the question into estimates for tube number, multiplicity, and union volume. Writing a line in terms of direction and transverse position led naturally to the action of $SE(n)$. The oriented line space was identified as $\mathscr{L}_n^+ \cong SE(n)/(\mathbb{R} \times SO(n-1))$, the geometric action was lifted to quasi-regular representations, and the X-ray transform was shown to satisfy $X\pi(g) = \widehat{\pi}(g)X$. This common framework explains the role of translation and rotation covariance in both dimensions. In the plane, Fourier analysis and the $SO(2)$ angular action organize the almost-orthogonality behind the classical L^2 method. In three dimensions, spherical harmonics and affine rescaling organize direction and scale, but they do not control positional and multiscale concentration. Representation theory therefore clarifies the structure of the problem without replacing its geometric estimates.

The same viewpoint extends naturally to higher dimensions, where $S^{n-1} \cong SO(n)/SO(n-1)$ and the oriented line space remains a homogeneous space of $SE(n)$. This suggests studying induced representations on line spaces, linearized tube-averaging operators, and connections with X-ray, Radon, Nikodym, and k -plane transforms. Affine covariance may also provide a useful language for comparing tube configurations across scales. Such a framework could help distinguish consequences of symmetry from the genuinely new non-concentration or incidence estimates required in higher dimensions. Nevertheless, the higher-dimensional Kakeya conjecture remains open, and no current representation-theoretic decomposition controls the nonlinear choice of tube positions across many scales. The realistic contribution of group representation theory is therefore not automatic proof, but a unified language in which the remaining geometric obstacles can be isolated and compared more precisely.

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