

Research Progress on Solving Partial Differential Equations Using Physics-Informed Neural Networks

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Abstract. Physics-Informed Neural Networks (PINNs) are a new method which, over the past few years, due to the development of the deep learning technology, has demonstrated significant potential in solving Partial Differential Equations (PDEs). Directly incorporating physical laws into the model training process does not only increase the generalization capability of a model but also minimizes its reliance on data as well as enhances the interpretability of a model. Nevertheless, PINNs continue to encounter many issues with PDEs with high dimensionality and nonlinear, chaotic dynamics including error propagation, absence of initial conditions, and big data. Thus, the aim of the review is to summarize and discuss the existing research developments in the field of solving PDEs with the help of PINNs, touching upon various components of the field, such as basic theories, as well as particular applications, and results of the research concerning the development of various optimization algorithms, network optimization, and boundary conditions.

Keywords: Physics-Informed Neural Networks, Partial Differential Equations, Initial Conditions, Deep Learning Technology, Boundary Conditions

1. Introduction

Partial Differential Equations (PDEs) are the basis of description of very diverse physical phenomena in science and engineering. Macroscopically flowing fluids and rotating heavenly bodies to microscopically diffusing particles and the diffusion of molecules, PDEs are the theoretical frameworks, giving a rigorous mathematical argument of what should be the case with these phenomena. An example is in fluid dynamics, where the Navier-Stokes equations are the fundamental tool to study the behaviour of fluids, which are used to simulate atmospheric circulation and study the flow of air past aircraft in aerospace engineering. Maxwell equations in electromagnetism are the unifying equations of electrical and magnetic phenomena, which provided the theoretical framework of the invention of the modern technologies of communication and power transmission. However, the task of solving PDEs is a very difficult one. In the last several decades, the Finite Element Method (FEM) [1], Finite Differentiation Method (FDM) [2] and Finite Volume Method (FVM) [3] have been used with major success applying the conventional numerical methods. But in practice, these methods have a number of disadvantages in the case of high dimensional spaces and nonlinear problems. When the dimensions are very high, the cost of computing is very high because the complexity is exponentially growing. Also, these techniques

normally demand full boundary conditions, initial conditions and correct equation parameters. In most practical scenarios, not all of this information is available, or may not be certain, and this puts a serious constraint on the conventional numerical solutions techniques. In highly nonlinear equations, iterative solution methods are frequently employed in the conventional numerical techniques, but the convergence rate of the iterative procedure is hard to determine. This can result in the process of trapping in local optimum solutions and this results in the computational results being very far away the actual values. As the computer technology rapidly advanced, and data-driven theories gained their suggestions, machine learning, and specifically the neural network technology, has introduced a new avenue to solve PDEs. Neural networks, its strong nonlinear mapping properties and data-fitting features are able to acquire complex patterns and laws using large volumes of data. Physics-Informed Neural Networks (PINNs) [4] employ the principles of the physical law in the training of a neural network and overcomes the disadvantage of conventional neural networks, which only adopt data-driven methods. In the case of PDEs, PINNs are also capable of using the full range of the information in data and are also able to follow physical laws strictly. They therefore portray benefits compared with the traditional approaches in such complicated situations like scarcity of data, extreme noise interference, or high nonlinearity of the problem. Indicatively, seismic waves, Chao Song et al. [5] suggested a seismic wave travel time tomography algorithm (PINNtomo) using PINNs which can also predict the wave velocities of shear waves and longitudinal waves of seismic waves. In this way, the speed of seismic waves can be better predicted in advance prior to the occurrence of an earthquake thus deducing when the earthquake will take place. Since the appearance of the idea of PINNs, the related studies have appeared as a steady stream, which has already resulted in some achievements in theoretical studies and in real applications. Theoretically, the mathematical basis of PINNs has been deeply studied by a range of scholars, with fundamental problems of convergence, stability, and the accuracy of the approximations of solutions of PDEs having been addressed. Mishra [6] and Molinaro demonstrated that given some regularity, PINNs are able to uniformly approximate the true solution of PDEs and summarized that the convergence rate of PINNs depends upon the depth and width of the network. Meng et al. [7] concentrated on the speed of the solution of PINNs, enhanced PINNs, and devised a technique known as PPINN. The solution pace of PINNs is enhanced by dividing the time dimension of PDEs into several smaller-problems and solving each sub-problem independently. PIINNs have been used successfully in various fields in the application field, including fluid mechanics, solid mechanics, materials science, electromagnetism, and medicine. Omarova et al. [8] used PINNs to solve the river silting problem in the field of fluid mechanics. The findings indicated that PINNs are able to produce a smoother curve in the fluctuation of pressure with respect to time as opposed to standard solution approaches and incur fewer costs in terms of computational time. Sarah Berkhahn and Matthias Ehrhardt [9] provided a comparison of the performance of PINNs and the results of the numerical method of Non-Standard Finite Differences (NSFD) in the medical sector, which allows predicting the time of the next outbreak of COVID-19 in the region. Nevertheless, even with such important advances, a great number of important questions concerning the connection between PINNs and the solutions of PDEs remain to be still explored. This article is an attempt to offer a thorough, methodical, and detailed survey of the research advances on the complexion between the solutions of PDEs and PINNs. We will start by taking a closer look at the underlying principles and structure of PINNs, and expound on the exact process of putting the physical information into the neural network training process. We shall then broadly categorize the status of application of PINNs in solving the various types of PDEs and we shall give an in-depth analysis of the outstanding achievements and challenges of the various fields. Moreover, we will

examine the major problems in the existing researches, outline the existing ways of improvement of PINNs, and discuss the future directions of the research with the hope that it will be referenced by researchers in other related studies.

2. Basic principles of Physics-Informed Neural Networks and their application in solving partial differential equations

Physics-Informed Neural Networks (PINNs) is a type of artificial intelligence learning that is a blend of physical learning knowledge and deep learning. Information neural networks are PINNs and the distinguishing feature of these neural networks is the addition of a physical information term, i.e. the laws of physics, to the loss function. It is possible to summarize their workflow into five steps: the identification of the physical problem and physical laws, the creation of a computation graph, the creation of a loss function and data preparation, the implementation of optimization strategies, and verification and evaluation of the model.

2.1. Determining the physical problem and physical laws

The initial step of the PINNs is to state the problem to be solved and the physical laws. As an illustration, in dealing with heat conduction problems, one deals with the heat conduction equation and the model construction is launched on the basis of the equation.

2.2. Constructing the computation graph

The second one is building a Computation Graph which is a graph structure that models mathematical calculations or program execution. It can automatically show the complicated process of calculation in the information neural network as well as precondition the later process like Automatic Differentiation (AD) [10]. The computation graph construction involves two stages in which the network architecture definition and AD configuration are defined. The first one is that a proper neural network structure is to be chosen, that is, with an input layer (to feed the spatial coordinates (x, t)) and an output layer (to emit the estimated numbers of the physical problem). Secondly, the calculation graph tracking mechanism must be set to allow high-order parts computation derivatives.

2.3. Building the loss function and preparing data

The third step involves building the loss function and preparing the data. Firstly, based on the forward or inverse problem of the PDE, a loss function is constructed that incorporates initial conditions, residuals, boundary conditions, and data errors. Such a loss function can be expressed as follows:

$$L = \lambda_{\text{PDE}}L_{\text{PDE}} + \lambda_{\text{BC}}L_{\text{BC}} + \lambda_{\text{IC}}L_{\text{IC}} + \lambda_{\text{Data}}L_{\text{Data}}$$

where λ represents the weight of each part of the loss according to the problem requirements, and each part of the loss function is expressed as follows

$$L_{PDE} = \frac{1}{N_r} \sum_{i=1}^{N_r} |N[\mu_\theta(x_i, t_i)] - s(x_i, t_i)|^2$$

$$L_{BC} = \frac{1}{N_{BC}} \sum_{i=1}^{N_{BC}} |\mu_\theta(x_i) - q(x_i)|^2$$

$$L_{IC} = \frac{1}{N_{IC}} \sum_{i=1}^{N_{IC}} |\mu_\theta(x_i, 0) - f(x_i)|^2$$

$$L_{Data} = \frac{1}{N_d} \sum_{i=1}^{N_d} |\mu_\theta(x_i, t_i) - u(x_i, t_i)|^2$$

In the above four functions, the content inside the absolute value represents the difference between the experimental predicted value and the observed value.

Secondly, sample points of the PDE solution are generated through temporal and spatial sampling, and data for boundary conditions and initial conditions are prepared simultaneously.

2.4. Implementing optimization strategies

The fourth step is to implement the optimization of the entire model. Generally, the Adam optimizer is used. Through back propagation, the gradient of the total loss with respect to the network parameters is calculated, and these parameters are continuously adjusted to minimize the loss function of the PDE.

2.5. Verifying and evaluating the model

The fifth step is to monitor the variation trend of the loss function during the solution process to determine whether it converges. If it does not converge, the computation graph, the weights of the loss function, etc., are readjusted.

For a complete PDE, it has the following form:

$$\begin{cases} u_{tt} = \alpha^2 u_{xx} & 0 < x < L, t > 0 & (1) \\ u|_{x=0} = 0, u|_{x=L} = 0 & & (2) \\ u|_{t=0} = \varphi(x), u_t|_{t=0} = \psi(x) & & (3) \end{cases}$$

where α in equation (1) refers to a physical parameter, representing the physical property corresponding to the equation to be solved by PINNs; equation (2) represents the boundary value condition of the PDE; and equation (3) represents the initial value condition of the PDE.

The figure below shows the workflow of a PINN, where μ_θ denotes the output value, and θ represents the value obtained under the parameters of this information neural network. In solving

PDEs, PINNs reduce the loss function $L(\theta)$, thereby improving the accuracy of the numerical solution. In the forward problem of solving PDEs using PINNs, the loss function can be expressed by the following formula:

$$L(\theta) = w_i L_i(\theta) + w_b L_b(\theta) + w_r L_r(\theta).$$

where w_i , w_b , w_r respectively represent the weights of the initial condition, boundary condition, and control equation function.

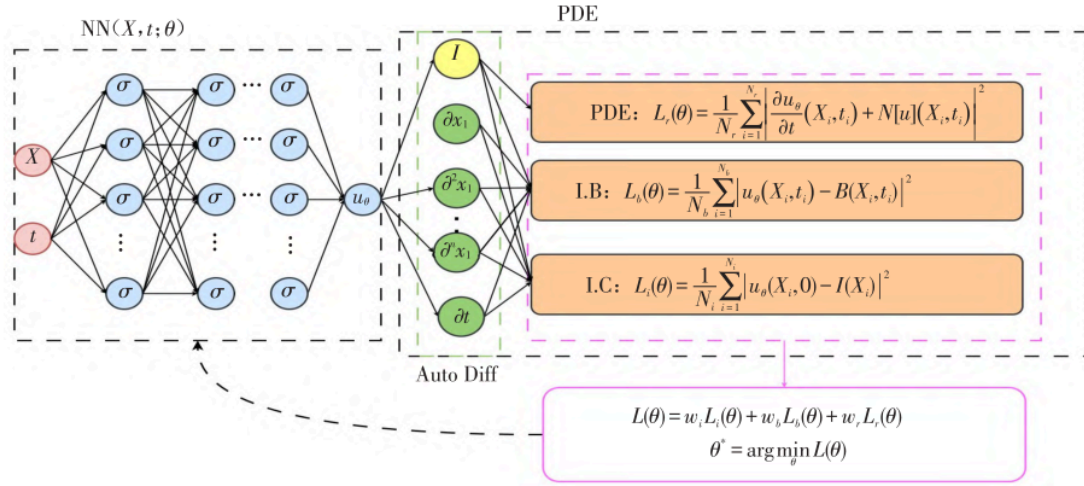


Figure 1. Schematic diagram of PINN structure [11] (Liu Xiaoting et al., 2024)

3. Applications of Physics-Informed Neural Networks in solving partial differential equations in different fields

3.1. Application in wave equations

The wave equation, also known as the wave equation, is a set of partial differential equations derived from Maxwell's equations that describe the wave characteristics of electromagnetic fields. It mainly describes various wave phenomena in nature. The general form of its one-dimensional equation is as follows:

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

where u is a scalar function of position x and time t , representing the displacement or amplitude of the wave.

Conventional numerical approaches for resolving the wave equation can typically be categorized into three distinct classes: the separation of variables technique [12], the characteristic curve methodology [13], and the Fourier transformation approach [14].

Separation of Variables Technique: The method is used to break down the multivariate function in a partial differential equation into a product of multiple univariate functions. The partial differential equation can be transformed to one or more ordinary differential equations through the replacement of this multiplication of a number of univariate functions with the wave equation. These are then

ordinary differential equations which are solved sequentially and superposition is then used to arrive at the solution of the partial differential equation. However, this method has limitations, in that it has the challenges of solving nonlinear or inhomogeneous wave equations.

Characteristic Curve Methodology: This method is a solution of the wave equation by exploiting the characteristic equations of the wave function. It finds a set of curves whereby the partial differential equation is reduced to ordinary differential equation. This is because the partial differential equation is then solved by the integration of the solutions of the equations on specific curves. However, this method proves to be effective only to the unidimensional wave equations; with increasing dimensionality, the method becomes impractical.

Fourier Transformation Approach: In this approach the resolution of the wave equation is transformed into the representation of a Fourier series expansion, the equation is resolved, and the solution of the wave equation is formulated by the use of inverse Fourier Transformation. However, due to the need to deprive Fourier transformation and its inverse process, the method contains certain errors, which leads to the loss of accuracy in the results.

As a result, in order to avoid the problems illustrated above, it is very appropriate to apply PINNs to solve wave equations [15]. Wu Danlan and others incorporated the formulation and parameters of the wave equation as subsystems of PINNs and then applied the Deep Neural Network (DNN) framework of PINNs to the learning of data. Following numerous training cycles, they were eventually capable of approximating the resolution of the wave equation. Through iterative experimentation, they discovered that when the quantity of concealed layers in the DNN is 5 and the quantity of neurons is 50, the deviation is minimized [16]. Fu Beibei employed PINNs and their enhanced algorithm, Gradient-enhanced Physics-Informed Neural Networks (gPINNs), to the forward and inverse resolution of unidimensional, bidimensional, and Tri dimensional wave equations respectively.

This research demonstrated that PINNs and their superior algorithms were possible and efficient, and offered a scientific and engineering foundation of computations [17]. Lu Yongjun used $\sin(\frac{\pi}{2}x)$ as the activation function to reduce the order of high-order wave equations and conducted numerical experiments. The experimental findings indicate that the approach is better than the conventional PINN approach that consumes fewer computational resources, shorter computation period of around 50 percent and better computational accuracy of 10^{-2} to 10^{-4} .

Overall, regardless of whether PINNs are utilized in the process of solving wave equations or enhanced PINNs are utilized instead, their performance is by far better than that of conventional numerical solutions to the problems.

3.2. Application in nonlinear Schrödinger equations

The nonlinear Schrodinger equation is a partial differential equation that is used to model the time dependence of the quantum state of a physical system. It predominately characterizes the systems that exhibit strong nonlinear effects and it can also characterize the nonlinear phenomenon in the field like fluid mechanics, nonlinear optics and optical fiber communications. It assumes a general shape as follows:

$$i\hbar \frac{\partial \varphi}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2 \varphi}{\partial x^2} + g|\varphi|^2 \varphi = 0$$

where $\varphi(x,t)$ represents the complex solution of the nonlinear Schrodinger equation, which is the quantum state of the system; m is the particle mass; g is the nonlinear coefficient, which is the property of the nonlinear term; and $\hbar$ is the reduced Planck constant. Conventional numerical solutions of this equation usually involve the mapping technique [18], Hirota technique [19], Darboux transformation technique [20], split-step Fourier technique [21] and pseudo spectral technique [22]. These conventional methods, however, have a disadvantage by involving cumbersome calculations, and are expensive to compute. Thus, PINNs, which have the ability to simplify calculation, are very appropriate in solving this equation.

The concept of transfer learning was offered by Yi Kai [23] to enhance the performance of PINNs in nonlinear Schrodinger equations. Various initial and boundary conditions were considered as different views to form a pre-trained nonlinear Schrodinger equation model, and thus they investigated its optimization impact on the solution of the nonlinear Schrödinger equations with the help of PINNs. The usefulness of the offered optimization method was confirmed in a great number of experiments [24]. Zhang Yu used PINNs in high-order, coupled, and variable-coefficient nonlinear Schrodinger equations in parallel and multi-parallel forms. This method successfully determined soliton solutions and rogue wave solutions, minimized data needs and errors, enhanced computational efficiency and flexibility as well as identified variable-coefficient inversion and parameter identification [25]. Wei-xing Qiu et al. added prior information as an additional term in the loss function to facilitate the distribution of objects in target positions to enhance the predictive power of PINNs. They also added adaptive weights and adaptive activation functions in PINNs. This facilitates the calculation operation to make a dynamic alteration of the weights in the function to maximize the prediction. The improved PINNs were used to find the soliton solutions of coupled nonlocal nonlinear Schrodinger equations, having two different soliton solutions of non-degeneracy and two different soliton solutions of degeneracy. The PINNs loss was optimized and paired with the transfer learning to enhance the performance of PINNs in solving the complex nonlinear Schrodinger equations (Chen et al. [26]). Pu et al. [27, 28] used PINNs to solve solutions of solitons, breather and rogue waves of nonlinear Schrodinger equations respectively. Huang et. al. [29] applied active learning to the training of PINNs in the learning of first-order rogue waves. The results showed that active learning can achieve higher convergence rate of PINNs and requires fewer data to obtain the required accuracy than the dynamic random sampling which is generally used in PINNs. Besides, they optimized PINNs by implementing an adaptive time-stepping method that is effective in solving the problem that PINNs do not directly deal with second-order rogue waves. Lastly, it can be concluded that in comparison to the standard methods of solving nonlinear Schrodinger equations, the use of PINNs and their improved tools can allow reducing the computational cost considerably. Besides that, PINNs can be applied to solve problems in the case of the limitation of the conventional solution processes. Therefore, with the introduction of PINNs and their improved tools, the cost of computation that people have to contend with in addressing this issue is greatly reduced.

3.3. Application in fluid mechanics

When discussing fluid mechanics, it is necessary to mention the Navier-Stokes equations [30]. These equations are physically explained by the fact that the force inertial of an element of a fluid, at a point in time and position, is equal to the sum of the external forces. These external forces are the pressure difference force that causes its motion, the viscous friction force that prevents its motion and the gravity. In the meantime, the mass remains constant in the process. The Navier-Stokes

equations for incompressible fluids consist of two equations, which can be specifically expressed as follows:

1) Momentum Equation

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho (\mathbf{u} \bullet \nabla) \mathbf{u} = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f}$$

where $\rho \frac{\partial \mathbf{u}}{\partial t}$ is the unsteady term; $\rho (\mathbf{u} \bullet \nabla) \mathbf{u}$ is the convective term; $-\nabla p$ is the pressure term; $\mu \nabla^2 \mathbf{u}$ is the viscous term; and $\mathbf{f}$ is the body force term.

2) Continuity Equation (Mass Conservation Equation)

$$\nabla \cdot \mathbf{u} = 0$$

Traditional numerical methods for solving the above equations include the finite element method, finite difference method, and finite volume method. However, these methods have strict requirements on initial conditions and boundary conditions. If these conditions change, re-computation is required, which leads to a significant increase in the computational load when using traditional numerical methods for solving, resulting in the lag of traditional methods in computation. Therefore, PINNs are suitable for addressing such problems.

Sun et al. [31] aimed to establish a model that can replace the Navier-Stokes equations. By directly integrating the laws of conservation of mass and momentum of fluids into the loss function, they were able to directly train the flow field model. Additionally, they took many scenarios in hemodynamics as examples and applied the PINN method to them. Finally, it was proved that the results of solving this equation using PINNs are consistent with those obtained using traditional numerical solutions. Raissi et al. [32] simulated the lift and drag of objects. On the basis of these equations, they further integrated the multi-displacement equation of object motion and the convection-diffusion equation into the loss function. Finally, the flow field was reconstructed through the observed data, and then information such as the lift of the object was reconstructed.

In addition to the Navier-Stokes equations, there are many other areas in fluid mechanics that can utilize PINNs. To improve the accuracy of hydrodynamic prediction for ship constraint models and facilitate the modeling of ship maneuvering motion, Zhao et al. [33] used the planar motion mechanism constraint model test data to couple the prediction of hydrodynamic forces and moments. The final research results showed that PINNs require less data than the original methods and have higher accuracy. Wang et al. [34] utilized the characteristic of PINNs that require a small amount of data to reconstruct experimental data with a small sample size, generate velocity fields, and predict pressure fields. Yu Hao [35] linked the problem of meniscus oscillation in capillary rise with PINNs. The results showed that PINNs can well predict the rising height and oscillation phenomenon of the liquid surface in capillaries, and both the capillary radius and contact angle have significant effects on the meniscus oscillation phenomenon.

In summary, the emergence of PINNs has made solving equations in fluid mechanics more convenient, overcoming the cumbersome processes and high computational costs of traditional solution methods.

4. Existing problems and improvements

4.1. Existing problems

Although PINNs are currently regarded as a more efficient and convenient solution method in many fields, they still have limitations. The current problems encountered by PINNs can be categorized into the following types:

Multi-objectivity and Imbalance of the Loss Function: Due to the large dimensions and orders of magnitude of the loss function of PINNs, the optimization process is prone to being dominated by one term, making it difficult for other terms to converge.

Problems with Computational Efficiency in High-Dimensional Problems: Since each node needs to solve high-order derivatives through automatic differentiation technology, once faced with high-dimensional problems, the computational load increases exponentially, leading to a rapid rise in computational costs.

Training Failure and Low Convergence Accuracy: The gradient of the PDE residual loss term with respect to the network parameter θ is smaller than that of the data fitting term. This causes the optimizer used to tend to minimize the error, resulting in deviation from the due accuracy of the data. Consequently, the final obtained data often fits the data points but does not satisfy the physical laws.

4.2. Derivatives and improvement methods based on PINNs

Since PINNs still have many problems in specific situations, many different types of PINNs have been derived in specific fields to specifically address a certain field. In 2021, Lu [36] and colleagues introduced an adaptive refinement approach (RAR) designed to enhance the training precision of PINNs. RAR dynamically modifies the boundary constraints throughout the training phase and elevates the training precision through the progressive addition of training samples until convergence toward the exact solution is achieved. Wang and colleagues [37] presented a learning rate decay strategy, representing an adaptive technique for modulating the weighting coefficients of loss components. Specifically, the weighting factors of the loss function are dynamically recalibrated throughout the computational procedure to enhance the predictive precision of PINNs. Huang [38] and collaborators constructed a Physics-Informed Neural Network framework (Log-SA-PINN) for modeling groundwater flow dynamics in the vadose zone utilizing the simulated annealing methodology. They optimized the annealing process using a logarithmic modification strategy and tested the results with those obtained using the traditional PINN. The results were relatively similar. Adaptive activation mechanism was presented by Jagtap et al. [39], that increases the predictive accuracy of PINNs by adjusting the activation function on the hidden layers. Guo and colleagues [40] came up with a Pre-trained Physics-Informed Neural Network (PT-PINN). Preliminary training is implemented on the small sub-problem domains in advance to the full problem domain and the pre-trained weights serve as the starting point during formal training, thereby increasing the accuracy and reliability of PINNs on large scale problems. Variational Physics-Informed Neural Network (VPINN) [41, 42] was developed on the basis of the Galerkin framework. It uses variational formulation of PINNs to define the loss of PINNs and then integrates by parts the loss of PINNs to reduce the order of the differentiated operators thereby improving the accuracy and reliability of PINNs predictions. These derivative models of PINNs are more accurate and reliable than the baseline PINNs in specific application area.

5. Conclusion and outlook

The article summarizes and examines the research progress of Physics-Informed Neural Networks (PINNs) in the field of solving Partial Differential Equations (PDEs), comprehensively explores theoretical background to technical implementation, and finally to the current challenges, developments and products. The concept of PINNs integrates physical laws and deep learning technology, which introduces a new method of solving PDEs. However, in practical cases of complex nonlinear PDEs and high dimensional problems, such problems as the high difficulty of the solution and the curse of dimensionality still persist. Future directions of the study to be pursued should involve development of more effective training procedures and adaptive refinement strategies that can increase the accuracy, precision, and efficiency of PINNs in finding solutions to complex PDEs. Furthermore, the connection between PINNs and technologies in other fields will only increase gradually in the future, and it will become the basis of technological progress.

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