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Quantum Communication of Images with 3-State Protocols

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Abstract. This paper investigates two distinct protocols and one combined protocol for image teleportation based on three quantum states. It begins by reviewing key linear algebraic concepts and formalized quantum states in Hilbert space. Building on this, it analyzes a variation of dense coding using non-maximally entangled (polluted) states. It demonstrates that while full distinguishability of four Bell-like states is lost under entanglement degradation, it is possible to design a restricted set of operations that maximizes distinguishability; it designs a new protocol of superdense coding based on three Bell states. In the next section of the article, it introduces a three-level energy system and designs a quantum teleportation protocol based on the three states. It also introduces the HSV color model, which features three parameters that are used to represent a color. With such a protocol, it can teleport a color with high confidence and security. Both of these protocols can be used for image teleportation technologies. Finally, it integrates the superdense coding protocol and the quantum teleportation protocol to create a highly theoretical image teleportation protocol that can perform filtering, encoding and decoding, and emergency destruction on colored images with enhanced security. An additional participant is added, so that the quantum state will be modified during the teleportation process by this new participant. All of these protocols can be applied in future image teleportation techniques.

Keywords: Dense Coding, Image Teleportation, Quantum Communication, Quantum Teleportation

1. Introduction

After quantum theories were published, Einstein, Podolsky, and Rosen questioned the completeness of quantum mechanics [1]. However, further applications of quantum mechanics has been developed decades later. Bennett and Wiensner laid the foundation for all quantum communication theories: they introduced dense coding with EPR pairs [2]. Quantum teleportation was then proposed by Bennett et al. in 1993 [3], and later demonstrated on a satellite link [4]. Theoretical theories of quantum computing were constructed by Barenco et al. [5]. Recent researches in secure quantum communication have been realized in integrated photonic systems [6], and high-dimensional teleportation has been achieved experimentally [7]. Some specific techniques such as the extensions of dense coding and teleportation to multipartite entanglement have been studied [8], and atomic qubits were used for dense coding in trapped-ion systems [9]. Dense coding was experimentally

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demonstrated by Mattle et al. [10], and secure direct communication protocols using high-dimensional superdense coding were proposed by Wang et al. [11].

This paper develops two theoretical quantum communication protocols that can be applied directly for image teleportation technologies using mathematical methods of linear algebra and functional analysis. It first introduces the fundamental concepts for quantum communication. It starts with the basics of linear algebra in Equation (3) and the Kronecker product in Equation (6). Then it discusses quantum states in Hilbert spaces. Building on that, it discusses the reduced density operator and POVM, one of the most important fundamental concepts for quantum teleportation in Equation (11). It also covers the classical superdense coding protocol which utilizes four Bell states in section. After a brief introduction of the fundamentals of this research, it moves on to two newly designed protocols. The first one is a variation of the original superdense coding protocol, which utilizes three Bell states instead of four. This protocol is more accurate than the classical one when delivering black-white-gray images (Section 3). It applies Helstrom's theorem to compare the two protocols and derives a mathematical expression for their accuracy (Theorem 1 and Theorem 2). Furthermore, it designs a numerical simulation of this image teleportation protocol and compares the result under various noisy environments (Figure 3). The second protocol is a quantum teleportation based on a 3-level energy system (Section 4). It introduces interdisciplinary concepts such as the HSV color model (Figure 4), which is a model that is commonly used in visual arts; it integrates these models from very distinct subjects: visual art and quantum physics, which allows it to decode the quantum state delivered to a specific color. This integrated protocol can be used for image teleportation with colors. These newly designed protocols, 3-state superdense coding, and HSV-quantum teleportation, can be used in image teleportation technologies while preserving high accuracy and confidentiality.

2. Preliminaries

This section introduces some basic facts and notions used in this paper.

2.1. Basics of matrices and the Kronecker product

A matrix is defined as an array or table of elements. An $n \times m$ matrix has n rows and m columns, and a_{ij} represents the element on the i th row, m th column. For example, the following matrix is a 2×3 matrix.

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 2 & e & \pi \end{pmatrix} \tag{1}$$

e is represented by a_{22}, while 4 is represented by $a_{1,3}$. A matrix composing of all real elements is a real matrix, while a matrix composing of all complex numbers is a complex matrix. A square matrix has n rows and n columns. A row matrix only has one row, while a column matrix only have 1 column. A diagonal matrix only has elements on its diagonal. I_n, the unit matrix is a special type of diagonal matrix such that all its elements are 1. Two similar matrices have the same rows and column numbers. The additions are applicable only when two matrices are similar.

The multiplication of a matrix is applicable only when the column number of the first matrix is the same as the row number of the second one. For example, only a matrix 2×3 can multiply a matrix 3×5, while a matrix 2×5 cannot.

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Let $A=(a_{ij})_{ms}$, $B=(b_{ij})_{sn}$. The resultant matrix $C=AB$ has m rows and n columns, and each element is given as:

$$c_{ij}=\sum_k^s a_{ik}b_{kj} \tag{2}$$

Where i ranges from 1 to m and j ranges from 1 to n. Here are some basic properties of matrix multiplication:

$$\begin{gathered}(AB)C=A(BC)\\ \lambda(AB)=(\lambda A)B\\ A(B+C)=AB+AC \qquad (B+C)A=BA+CA\\ I_nA=AI_n=A\end{gathered} \tag{3}$$

In most cases, please note that $AB\neq BA$.

The transverse operation of a matrix is an operation that flips the matrix across its diagonal by switching the columns and rows. A matrix $A=(a_{ij})_{mn}$ transverses into $A^T=(a_{ji})_{nm}$, therefore, taking the transverse of a matrix $n\times m$ will give us a matrix $m\times n$. An interesting property about the transverse is that when taking the transverse of the product of two matrices,

$$(AB)^T=B^TA^T \tag{4}$$

Except for the multiplication of two matrices, there is another operation that allows us to combine two matrices, which is the Kronecker product ⊗ (or the tensor product). The Kronecker product is defined as:

$$A\otimes B=\begin{bmatrix} a_{1,1}B & a_{1,2}B & a_{1,3}B & \dots & a_{1,n}B\\ a_{2,1}B & a_{2,2}B & a_{2,3}B & \dots & a_{2,n}B\\ & & & & \\ a_{n,1}B & a_{n,2}B & a_{n,3}B & \dots & a_{n,n}B\end{bmatrix} \tag{5}$$

Unlike matrix multiplication, the Kronecker product directly integrates the two matrices, instead of taking summation. In fact, this preserves more information about both matrices. Here are certain properties of this operation:

$$\begin{gathered}(A\otimes B)(C\otimes D)=(AC)\otimes(BD)\\ A\otimes(B+C)=A\otimes B+A\otimes C\\ (A+B)\otimes C=A\otimes C+B\otimes C\\ (A\otimes B)\otimes C=A\otimes(B\otimes C)\\ (A\otimes B)^T=A^T\otimes B^T\\ (A\otimes B)^\dagger=A^\dagger\otimes B^\dagger\\ det(A\otimes B)=det(A)^m\cdot det(B)^n\\ Tr(A\otimes B)=Tr(A)\cdot Tr(B)\end{gathered} \tag{6}$$

Considering it through a tensor prospective, a (k,l) tensor T is the tensor product of k vectors and l 1-forms, noted as

$$T=T^{i_1,i_2\dots i_k}_{j_1,j_2\dots j_k}e_{i_1}\otimes e_{i_2}\dots\otimes e_{i_k}\otimes e^{i_1}\otimes e^{i_2}\dots\otimes e^{i_k} \tag{7}$$

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Noted that a tensor is a multi-linear mapping that maps the given space to R, which combines all the different basis and dual vectors. The linearity made this operation especially suitable in the scenario of quantum mechanics, and hence we can apply this operation when creating a complicated quantum state considering multiple systems.

2.2. Fundamentals of quantum mechanics

Consider a wave function $|\psi\rangle_{AB}$ in a Hilbert space $\mathscr{H}_{AB}$, where $\mathscr{H}_{AB}$ consists of two sub-Hilbert spaces, noted as $\mathscr{H}_A$ and $\mathscr{H}_B$, such that $\mathscr{H}_{AB} = \mathscr{H}_A \otimes \mathscr{H}_B$.

Hence, $|\psi\rangle_{AB}$ can be represented by the tensor product of a pair of orthogonal basis from these two subspaces, $|\alpha_i\rangle$ and $|\beta_j\rangle$.

$$|\psi\rangle_{AB} = \sum_{i,j} a_{i,j} |\alpha_i\rangle \otimes |\beta_j\rangle \tag{8}$$

For example, consider the two quantum states representing the spin of two subsystems:

$$|0\rangle = \binom{1}{0}, |1\rangle = \binom{0}{1} \tag{9}$$

A possible wave function composed by these two orthogonal basis could be:

$$|\psi^+\rangle_{AB} = \frac{\sqrt{2}}{2}|00\rangle + \frac{\sqrt{2}}{2}|11\rangle \tag{10}$$

This state is also entangled. This means that if one of the quantum states of one of the particles is measured, then the state of the other particle can also be predicted. The two possible product states here are $|00\rangle$ and $|11\rangle$. If the state of the first particle is $|0\rangle$, then the other particle must also be $|0\rangle$, as other possibilities such as $|01\rangle$ are impossible. This combined state indicates information of both systems(the spin of both particles) at the same time. Instead of having a traditional "vector" quantum state, this quantum state indicates multiple systems, hence it is considered a tensor or a matrix in this case.

An orthogonal projection measuring operator M_a defined on the space $\mathscr{H}_A$ is considered, which measures an observable in the system. Here$M_a$is defined as a POVM (positive operator-valued measure) operator $|\psi_j\rangle\langle\psi_j|$ such that $\sum_{j=1}^{n} |\psi_j\rangle\langle\psi_j| = I_n$. In order to extend the measurement to the entire Hilbert space, the tensor product of M_a is taken with a unit matrix$I_n$on the ancilla space B . It gives us a POVM $M_a \otimes I_n$ on the bipartite Hilbert space $\mathscr{H}_{AB}$. Hence the expected value of M_a can be presented by the following expression,

$$\begin{gathered}\langle M_a\rangle = \langle\psi|_{AB} M_a \otimes I_n |\psi\rangle_{AB} \\ = \sum_{i,j} a_{i,j}^* \langle\alpha_i, \beta_j|(M_a \otimes I_n) \sum_{h,l} a_{k,l} |\alpha_k, \beta_l\rangle \\ = \sum_{i,j,k} a_{i,j}^* a_{k,j} [\langle\alpha_i|_A M_a|\alpha_k\rangle] \\ = tr(M_a \sum_{i,j,k} |\alpha_k\rangle\langle\alpha_i| a_{i,j}^* a_{k,j}) \\ = tr(M_a \rho_a).\end{gathered} \tag{11}$$

In the above computation, it has used the fact that $|\beta_j\rangle$ is an orthonormal basis, leaving us with a more simplified expression. Further, ρ_a represents the reduced density operator, defined as

$$\rho_a = tr_B(|\psi\rangle\langle\psi|) = \sum_j (I_A \otimes \langle\beta_j|)|\psi\rangle\langle\psi|(I_A \otimes |\beta_j\rangle) = \sum_{i,j,k} a_{i,j}^* a_{k,j} |\alpha_k\rangle\langle\alpha_i|. \tag{12}$$

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Here, tr_B is defined as the partial trace, which means that we only take the trace of the indexes of space $\mathscr{H}_B$. In this case, the trace of all β indexes that represents the basis $|\beta_j\rangle \in \mathscr{H}_B$ is taken. The density matrix here is "reduced", since we take the trace of an index, reducing an index.

Note that a quantum state has to be square-normalized, which means $\int|\psi|^2 = 1$. Hence, applying this condition, it can solve for the coefficients. Simplifying the scenario, the measurement on the entire system instead of the subsystem $\mathscr{H}_a$ is considered, so that the POVM becomes a mapping from the reduce density matrix before the measurement to the reduced density matrix after the observation. Each term of the summation can be written as a mapping from the original reduced density matrix to a density matrix after observation:

$$\rho \to \frac{|\alpha_i\rangle\langle\alpha_i|\ \rho\ |\alpha_i\rangle\langle\alpha_i|}{tr(|\alpha_i\rangle\langle\alpha_i|\rho|\alpha_i\rangle\langle\alpha_i|)} = M_i \tag{13}$$

Where $tr(|\alpha_i\rangle\langle\alpha_i|\rho|\alpha_i\rangle\langle\alpha_i|)$ represents the probability of obtaining the value M_i. The operators $|\alpha_i\rangle\langle\alpha_i|$ constitutes a set of operators for POVM, such that $\sum|\alpha_i\rangle\langle\alpha_i| = I_n$.

2.3. Basics of quantum computation: dense coding

Super-dense coding makes use of the quantum entanglement of an EPR pair. For example, an EPR state is prepared to be

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{14}$$

Here it can obtain the state by applying the CNOT gate and the Hadamard gate H on the initial product state $|00\rangle$, such that $CNOT(H \otimes I)|00\rangle = |\Phi^+\rangle$. The sender Alice wants to send two classical bits that to the receiver Bob. Then Alice can apply one of four unitary transformations, namely the Pauli gates on the particle of the EPR pair she has. Table 1 gives the list of unitary transformations.

Table 1. Bell states and unitary transformation

quantum state after modification(Bell states)	Operation	Classical Qubit
$\vert\Phi^+\rangle$	$I \otimes I\vert\Phi^+\rangle$	00
$\vert\Phi^-\rangle$	$\sigma_z \otimes I\vert\Phi^+\rangle$	10
$\vert\Psi^+\rangle$	$\sigma_x \otimes \vert\Phi^+\rangle$	01
$\vert\Psi^-\rangle$	$i\sigma_y \otimes \vert\Phi^+\rangle$	11

After performing the Pauli gate, Alice will send her particle to Bob. Noting that the current quantum state has been affected by a series of gates in order to become one of four Bell states. The inverse of the initial transformation is applied, which is

$$B = (CNOT(H \otimes I))^{-1} = (H \otimes I)CNOT. \tag{15}$$

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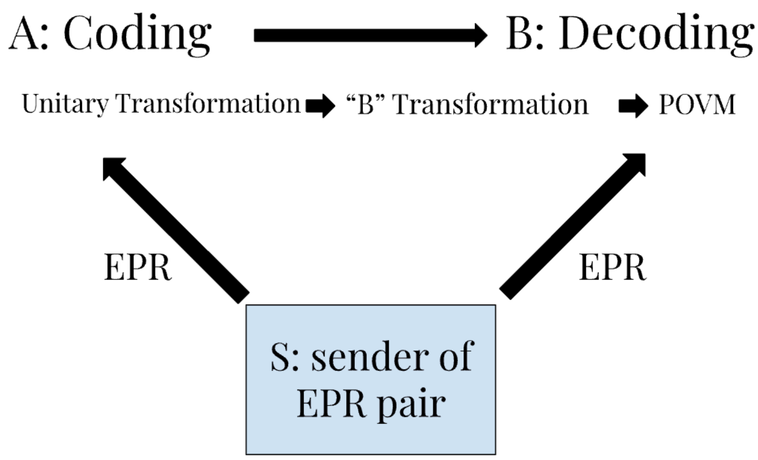

Figure 1. Quantum super dense coding

As shown in Figure 1, after the transformation, some POVM are applied to obtain the classical message that Alice wants to send. Note that the properties of POVM allow the communication process to have a high level of confidentiality. The reduced density matrix of system B of each Bell state is always $\frac{1}{2}I_n$. Hence unless the stalker captures both particles at the same time, it is impossible to obtain the message from Alice, as a partial trace is being taken on the quantum states, which leads to the loss of information and it is irreversible. However, as Alice sends her particle to Bob, POVM is applied using a set of operators. The POVM set of operators here is $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$, and the sum of the operators, as expected, is I_2. So Bob has two particles that contain the qubit Alice wants to deliver. Hence two POVMs are applied on the quantum states in order to derive the classical messages stored in the quantum state. Actually, it will get

$$\mathrm{POVM}_1 \rightarrow \begin{array}{ccccccc} O_1 & \rightarrow & |00\rangle & \overset{\mathrm{POVM}}{\rightarrow} & 0 & \rightarrow & |00\rangle \\ & \searrow & |01\rangle & & 1 & \rightarrow & |01\rangle \\ I_1 & \rightarrow & |10\rangle & \overset{\mathrm{POVM}}{\rightarrow} & 0 & \rightarrow & |10\rangle \\ & \searrow & |11\rangle & & 1 & \rightarrow & |11\rangle \end{array} \tag{16}$$

Thus, with two consecutive POVM measurements, it can obtain the classical bit from the quantum state with 100% accuracy, which allows us to decode Alice's message.

As Alice changes her own EPR, Bob's EPR also changes. The value of an observable is always fixed before observation of the observable in classical physics, whereas quantum mechanics allows us to achieve such a method of conveying information with high confidentiality.

3. Variation of dense coding using polluted entangled states with 1 EPR pair

A bipartite maximally entangled state ψ_{AB} is defined as a quantum state such that for the quantum state of each subsystem, $\psi_A = \psi_B = I/d$. Where d represents the dimension of the Hilbert space

For example, the bell state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is maximally entangled, hence for the two possible outcomes, $|00\rangle$ and $|11\rangle$, the possibility is $(\frac{1}{\sqrt{2}})^2 = \frac{1}{2}$, where 2 is the dimension of the Hilbert space as there are two particles. The four bell states, being maximally entangled, are orthogonal to each other. Theoretically speaking, superdense coding can achieve a 100% accuracy rate with such property. However, quantum states in the real world are often affected by multiple factors; these

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quantum states are not as ideal as bell states because they are not maximally entangled. However, we may generalize all 2-qubit quantum states in the following form.

$$|\beta\rangle = cos(\theta)|u_0, v_0\rangle + sin(\theta)|u_1, v_1\rangle, \tag{17}$$

where $|u_i\rangle$ and $|v_i\rangle$ are orthonormal bases such that $\langle u_i|u_j\rangle = \delta_{ij}$ and $\langle v_i|v_j\rangle = \delta_{ij}$ and $\theta \in [0, \pi/4]$.

The quantum state $|\beta\rangle$ can be prepared using the following operation: $CNOT(H(\theta) \otimes I_2)$, where $H(\theta)$ is defined by

$$(\begin{matrix} cos(\theta) & sin(\theta) \\ sin(\theta) & -cos(\theta) \end{matrix}) \tag{18}$$

Lemma 1. *Suppose* $u_0 = (\begin{matrix} cos(\gamma) \\ sin(\gamma)e^{i\delta} \end{matrix}), u_1 = (\begin{matrix} sin(\gamma) \\ -cos(\gamma)e^{i\delta} \end{matrix})$. *Then* u_0 *and* u_1 *form an orthonormal basis.*

Proof. Since $\langle u_0|u_1\rangle = 0$ and $\langle u_0|u_0\rangle = \langle u_1|u_1\rangle = 1$, these two vectors are orthonormal.

Furthermore, a function $E(\theta) = -cos^2(\theta)log_2cos^2(\theta) - sin^2(\theta)log_2sin^2(\theta)$ is defined to describe the entanglement of a quantum state. Note that when $\theta = \pi/4$, $E(\theta) = 1$, which is the maximum of the function. Thus, the state is maximally entangled at $\pi/4$.

However, as mentioned, the quantum states in the real world may be polluted. If the same four operations are applied to the quantum state, the four states will no longer be orthogonal to each other.

3.1. Discrimination of 4 states

Lemma 2 (Helstrom Bound). *Let* ρ_1 *and* ρ_2 *be two quantum states on a Hilbert space, with prior probabilities* p_1 *and* $p_2 = 1 - p_1$. *The maximum probability of correctly identifying the state using any POVM measurement is given by:*

$$P_{correct} = \frac{1}{2}(1 + \|p_1\rho_1 - p_2\rho_2\|_1) \tag{19}$$

where $\|A\|_1 = Tr[\sqrt{A^\dagger A}]$.

In the special case where $\rho_1 = |\psi_1\rangle\langle\psi_1|$ *and* $\rho_2 = |\psi_2\rangle\langle\psi_2|$ *are pure states, and* $p_1 = p_2 = \frac{1}{2}$, *the formula reduces to:*

$$P_{correct} = \frac{1}{2}(1 + \sqrt{1 - |\langle\psi_1|\psi_2\rangle|^2}). \tag{20}$$

Theorem 1. *We have*

$$\begin{aligned} (I_2 \otimes I_2)|\phi(\theta)\rangle &= cos(\theta)|00\rangle + sin(\theta)|11\rangle \\ (\sigma_x \otimes I_2)|\phi(\theta)\rangle &= sin(\theta)|01\rangle + sin(\theta)|10\rangle \\ (i\sigma_y \otimes I_2)|\phi(\theta)\rangle &= sin(\theta)|01\rangle - sin(\theta)|10\rangle \\ (\sigma_z \otimes I_2)|\phi(\theta)\rangle &= cos(\theta)|00\rangle - sin(\theta)|11\rangle \end{aligned} \tag{21}$$

The probability of correctly differentiating the 4 quantum states is

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$$P_{correct} = \frac{1}{2}(1 + sin(2\theta)). \tag{22}$$

Proof. Noting that the first quantum state is not orthogonal to the fourth quantum state, and the second state is not orthogonal to the third one, we may not fully differentiate these two pairs of quantum states, such that the accuracy of the measurement would be lower than 100% and greater than 50% since the first state and the fourth state are orthogonal to the second state the the third state.

Consider a reduced density matrix ρ such that $\rho = \sum p_i \sigma_i$, and σ_i are not orthogonal states. Recall that the probability of obtaining the i th result from POVM E_i is $tr(E_i \rho)$. Since ρ consists of a set of different states, the probability of obtaining the i th result with the j th state given is $tr(E_i \sigma_j)$. Hence, the probability of correctly obtaining the result (given the state σ_i and obtaining the state σ_i) is given by:

$$P_{correct} = \sum_i p_i tr(E_i \sigma_i) \tag{23}$$

However, since there are two confusable pairs: 1 and 4, 2 and 3, it would be slightly more complicated for us to design a POVM to achieve maximum accuracy. Thus, we consider the Helstrom bound for each confusable pair and analyze the maximum accuracy.

$$P_{correct} = \frac{1}{2}(1 + \sqrt{1 - |\langle\psi_1|\psi_2\rangle|^2}). \tag{24}$$

Noted that the inner product of the first state and the fourth state and the second state and the fourth state is given by:

$$\begin{aligned} \langle\psi_1 \mid \psi_4\rangle &= \left(\cos\theta\,\langle 00 \mid + \sin\theta\,\langle 11 \mid\right)\left(\cos\theta\,|00\rangle - \sin\theta\,|11\rangle\right) \\ &= \cos^2\theta - \sin^2\theta \\ &= \cos(2\theta) \\ \left|\langle\psi_1 \mid \psi_4\rangle\right|^2 &= \cos^2(2\theta) \end{aligned} \tag{25}$$

$$\begin{aligned} \langle\psi_2 \mid \psi_3\rangle &= \left(\cos\theta\,\langle 10 \mid + \sin\theta\,\langle 01 \mid\right)\left(\cos\theta\,|10\rangle - \sin\theta\,|01\rangle\right) \\ &= \cos^2\theta - \sin^2\theta \\ &= \cos(2\theta) \\ \left|\langle\psi_2 \mid \psi_3\rangle\right|^2 &= \cos^2(2\theta) \end{aligned} \tag{26}$$

For each confusable pair, the probability of correct distinguishing the pair is given by:

$$P = \frac{1}{2}(1 + sin(2\theta)) \tag{27}$$

Hence the probability of correctly distinguishing the 4 quantum states is

$$P_{correct} = \frac{1}{2}(1 + sin(2\theta)). \tag{28}$$

3.2. Discrimination of three states

Since it is impossible to fully distinguish all four quantum states due to the lack of orthogonality, we want to design three different operations to see if they can produce three states that are orthogonal to

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each other. Recall that for any arbitrary unitary matrix, it can be written as:

$$U = \begin{pmatrix} cos(a) & e^{ib}sin(a) \\ e^{ic}sin(a) & -e^{i(b+c)}cos(a) \end{pmatrix} \tag{29}$$

Thus, the three new operations that we design are

$$\{U_1 \otimes I_2, U_2 \otimes I_2, I_2 \otimes I_2\} \tag{30}$$

As these operations are applied on the initial polluted quantum state $cos(\theta)|00\rangle + sin(\theta)|11\rangle$, it can obtain three different quantum states

$$\beta = \begin{pmatrix} cos(a)cos(\theta) \\ e^{ib}sin(a)sin(\theta) \\ e^{ic}sin(a)cos(\theta) \\ -e^{i(b+c)}cos(a)sin(\theta) \end{pmatrix}, \quad \gamma = \begin{pmatrix} cos(d)cos(\theta) \\ e^{if}sin(d)sin(\theta) \\ e^{ig}sin(d)cos(\theta) \\ -e^{i(f+g)}cos(d)sin(\theta) \end{pmatrix}, \quad \delta = \begin{pmatrix} cos(\theta) \\ 0 \\ 0 \\ sin(\theta) \end{pmatrix}. \tag{31}$$

In order to simplify the case, we choose $cos(a) = 0, sin(a) = 1$ and $cos(d) = 0, sin(d) = 1$ so that the third quantum state is always orthogonal to the first two. The inner product of the three quantum states is taken, which gives the following equation:

$$e^{i(f+b)}sin^2(\theta) + e^{i(g+c)}cos^2(\theta) = 0. \tag{32}$$

Since both $sin^2\theta$ and $cos^2\theta$ are greater than zero, and $\theta \in (0, \frac{\pi}{4})$, $sin^2(\theta) \neq cos^2(\theta)$, the equation technically have no roots, hence fully differentiating three quantum states is impossible as well. However, we can still attempt to design a new way of superdense coding to achieve relatively high accuracy.

The sum of the inner product of the three quantum states is considered

$$|\langle\beta|\gamma\rangle|^2 + |\langle\beta|\delta\rangle|^2 + |\langle\delta|\gamma\rangle|^2 = \lambda(a, b, c, d, f, g). \tag{33}$$

While λ can be written as

$$\begin{gathered} \lambda(a,b,c,d,f,g) \\ = |sin^2(\theta)e^{i(b+f)}[sin(a)sin(d) + e^{i(c+g)}cos(a)cos(d)] \\ +cos^2(\theta)[cos(a)cos(d) + e^{i(c+g)}sin(a)sin(d)]|^2 \\ +|cos^2(\theta)cos(a) - cos(a)sin^2(\theta)e^{i(b+c)}|^2 \\ +|cos^2(\theta)cos(d) - cos(d)sin^2(\theta)e^{i(f+g)}|^2, \\ = |sin^2(\theta)e^{i(b+f)}[sin(a)sin(d) + e^{i(c+g)}cos(a)cos(d)] \\ +cos^2(\theta)[cos(a)cos(d) + e^{i(c+g)}sin(a)sin(d)]|^2 \\ +cos(a)^2|cos^2(\theta) - sin^2(\theta)e^{i(b+c)}|^2 \\ +cos(d)^2|cos^2(\theta) - sin^2(\theta)e^{i(f+g)}|^2, \end{gathered} \tag{34}$$

where$\theta \in (0, \frac{\pi}{4})anda, d \in [0, \pi/2], b, c, f, g \in [0, 2\pi)$.

Lemma 3. *The absolute minimum of* $\lambda(a, b, c, d, f, g)$ *is* $cos^2(2\theta)$.

Proof. Observe that by setting $a = \frac{\pi}{2}$ and $d = \frac{\pi}{2}$, the expression of λ can be greatly simplified. Since $cos(\pi/2) = 0$, the second and third term vanishes, which leads the following expression:

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$$\lambda = |sin^2(\theta)e^{i(b+f)} + cos^2(\theta)e^{i(c+g)}|^2 \tag{35}$$

Since both $sin^2(\theta)$ and $cos^2(\theta)$ are positive real numbers, it is obvious that the minimum function occurs when $e^{i(b+f)}$ and $e^{i(c+g)}$ have opposite signs. Hence, let $e^{i(b+f)} = 1$ and $e^{i(c+g)} = -1$, and the function becomes:

$$\lambda \geq |sin^2(\theta) - cos^2(\theta)|^2 = cos^2(2\theta) \tag{36}$$

Under the constraints $a = \frac{\pi}{2}$ and $d = \frac{\pi}{2}$, it is observed that the minimum of the function occurs when $b + f = 0$ and $c + g = \pi$, and the absolute minimum of the function $\lambda_{min} = cos^2(2\theta)$.

Furthermore, if a constraint is dropped, whereas only $a = \frac{\pi}{2}$, and d is still a variable, it can simplify the expression for λ, which gives the following:

$$\lambda = sin^2(d) \cdot |sin^2(\theta)e^{i(b+f)} + cos^2(\theta)e^{i(c+g)}|^2 + cos^2(d) \cdot |cos^2(\theta) - sin^2(\theta)e^{i(f+g)}|^2 \tag{37}$$

Using a similar approach, let $e^{i(f+g)} = 1$, $e^{i(b+f)} = -1$, and $e^{i(c+g)} = 1$, the expression becomes:

$$\begin{aligned} \lambda \ &= sin^2(d) \cdot |cos^2(\theta) - sin^2(\theta)|^2 + cos^2(d) \cdot |cos^2(\theta) - sin^2(\theta)|^2 \\ &= |sin^2(\theta) - cos^2(\theta)|^2 = cos^2(2\theta) \end{aligned} \tag{38}$$

$$\begin{aligned} \lambda = sin^2(d) \cdot |cos^2(\theta) - sin^2(\theta)|^2 + cos^2(d) \cdot |cos^2(\theta) - sin^2(\theta)|^2 \\ = |sin^2(\theta) - cos^2(\theta)|^2 = cos^2(2\theta) \end{aligned} \tag{39}$$

The final result obtained with one constraint is the same as that obtained with two constraints.

However, the expression $\lambda_{min} = cos^2(2\theta)$ might be the absolute minimum of the function, but a different constraint is going to be applied for verification.

Instead of $a = \frac{\pi}{2}$, we consider $a = 0$ as a different constraint. After simplification, the expression yields the following:

$$\begin{aligned} \lambda \ &= |cos(d)(sin^2(\theta)e^{i(b+f+c+g)} + cos^2(\theta))|^2 + |cos^2(\theta) - sin^2(\theta)\, e^{i(b+c)}|^2 \\ &\quad + cos^2(d)|cos^2(\theta) - sin^2(\theta)\, e^{i(f+g)}|^2 \\ &= cos^2(d)[|sin^2(\theta)e^{i(b+f+c+g)} + cos^2(\theta)|^2 + |cos^2(\theta) - sin^2(\theta)e^{i(f+g)}|^2] + \\ &\quad |cos^2(\theta) - sin^2(\theta)e^{i(b+c)}| \end{aligned} \tag{40}$$

Noted that the expression following $cos^2(d)$ in the first term is a positive real number, in order to minimize the function, $cos(d)$ must be 0, hence the expression for λ becomes:

$$\lambda = |cos^2(\theta) - sin^2(\theta)e^{i(b+c)}| \tag{41}$$

To minimize the expression, $e^{i(b+c)} = 1$, and the simplified expression yields:

$$\lambda_{min} = cos^2(2\theta) \tag{42}$$

Noted that under different constraints, it obtains the same result for the minimum value of λ_{min}, which is always $cos^2(2\theta)$.

Furthermore, if the set of values that allows this minimum to be achieved is chosen, the three new quantum states in this 3-state protocol can be obtained.

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The following values are chosen.

$$a = d = \pi/2, b + f = 0, c + g = \pi \tag{43}$$

Substituting it into the definition of the three states β, γ, δ, it obtains:

$$\begin{gathered} |\delta\rangle = (I \otimes I)|\phi(\theta)\rangle = cos(\theta)|00\rangle + sin(\theta)|11\rangle \\ |\beta\rangle = (U_1 \otimes I)|\phi(\theta)\rangle = cos(\theta)e^{ic}|10\rangle + sin(\theta)e^{ib}|01\rangle \\ |\gamma\rangle = (U_2 \otimes I)|\phi(\theta)\rangle = cos(\theta)e^{ig}|10\rangle + sin(\theta)e^{if}|01\rangle \end{gathered} \tag{44}$$

Since the individual value of b, f and c, g don't effect the accuracy. Let $b = \pi, f = -\pi$ and $c = 0, g = \pi$, hence U_1 and U_2 and I has the following expression:

$$U_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{45}$$

$$U_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence the three quantum states has the following expression:

$$\begin{gathered} |\delta\rangle = cos(\theta)|00\rangle + sin(\theta)|11\rangle \\ |\beta\rangle = cos(\theta)|10\rangle - sin(\theta)|01\rangle \\ |\gamma\rangle = -cos(\theta)|10\rangle - sin(\theta)|01\rangle \end{gathered} \tag{46}$$

With these three newly designed states, it can compare the inner product of these three quantum states. It obtains:

$$\begin{gathered} \langle\delta \mid \beta\rangle = cos(\theta)\langle 00 \mid 10\rangle - sin(\theta)\langle 11 \mid 01\rangle = 0 \\ \langle\delta \mid \gamma\rangle = -cos(\theta)\langle 00 \mid 10\rangle - sin(\theta)\langle 11 \mid 01\rangle = 0 \\ \langle\beta \mid \gamma\rangle = (cos(\theta)\langle 10| - sin(\theta)\langle 01|)(-cos(\theta)|10\rangle - sin(\theta)|01\rangle) \\ = -cos^2(\theta)\langle 10 \mid 10\rangle + sin^2(\theta)\langle 01 \mid 01\rangle = -cos^2(\theta) + sin^2(\theta) \\ = -cos(2\theta) \end{gathered} \tag{47}$$

So, the squared overlap is:

$$|\langle\beta \mid \gamma\rangle|^2 = cos^2(2\theta) \tag{48}$$

However, although a new protocol that is theoretically optimal has been designed, its precision has not been examined yet.

Theorem 2. *The maximum accuracy of the 3-state protocol is given by* $P_{correct} = \frac{2}{3} + \frac{1}{3} sin(2\theta)$

.

Proof. Recall the formula for correctly distinguishing the four pairs:

$$P_{correct} = \sum_i p_i tr(E_i \sigma_i) = \sum_i p_i \times Probability\ of\ correctly\ measuring\ \sigma_i \tag{49}$$

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Note that there is only one confusable pair: β and γ , so if the correct quantum state is δ, it can fully differentiate it from the other two states. Hence the probability of correctly measuring δ_i is 1.

Speaking of the confusable pair, the inner product is $cos(2\theta)$. It can apply the Helstrom bound:

$$P_{Helstrom} = \frac{1}{2}\left(1 + \sqrt{1 - cos^2(2\theta)}\right) = \frac{1}{2}\left(1 + sin(2\theta)\right) \tag{50}$$

Hence the probability of correctly measuring β and γ is $\frac{1}{2}\left(1 + sin(2\theta)\right)$. Hence,

$$P_{correct} = \frac{1}{3} \times 1 + \frac{1}{3} \times 2 \times \frac{1}{2}\left(1 + sin(2\theta)\right) = \frac{2}{3} + \frac{1}{3} sin(2\theta) \tag{50}$$

Thus, it can get derived the accuracy of the classical 4-state protocol and the 3-state protocol.

$$\begin{cases} 3 - stateprotocol: & P^{(3)}_{correct} = \frac{2}{3} + \frac{1}{3} sin(2\theta) \\ 4 - stateprotocol: & P^{(4)}_{correct} = \frac{1}{2}\left(1 + sin(2\theta)\right) \end{cases} \tag{51}$$

Where $\theta \in (0, \frac{\pi}{4}]$.

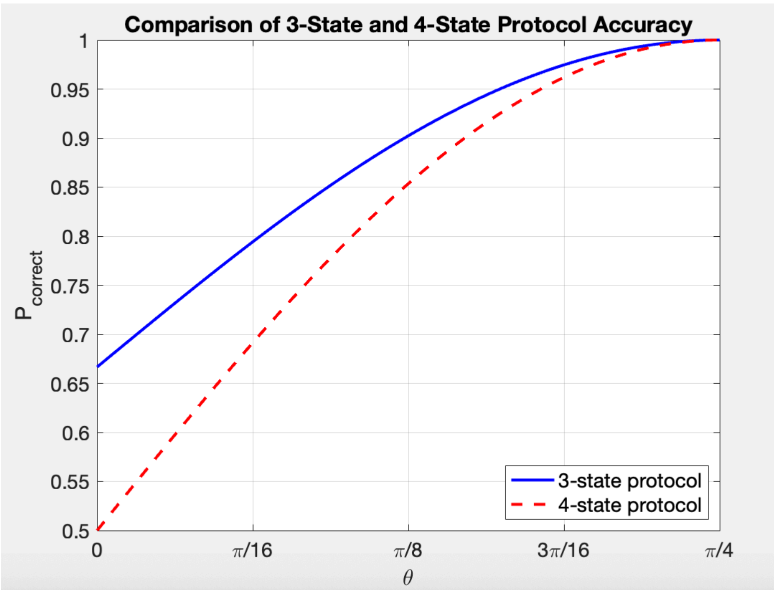

Figure 2. 3-state discrimination accuracy VS 4-state discrimination accuracy

In Figure 2, the red line represents the accuracy of the 3-state protocol while the blue line represents the 4-state protocol. It can be observed that both states achieve 100% accuracy at $\frac{\pi}{4}$, but the 3-state protocol has a higher accuracy for $(0, \frac{\pi}{4})$.

In conclusion, the 3-state protocol carries less information, but has a higher accuracy.

3.3. Comparison between the two protocols

To visualize the practical impact of quantum distinguishability limits, image transmission is simulated using both the 3-state and 4-state superdense coding protocols. A synthetic grayscale image is constructed using three levels—white, gray, and black—corresponding to quantum states 0, 1, and 2, with the accuracy given by the theoretical bounds. The results are shown in Figure 3.

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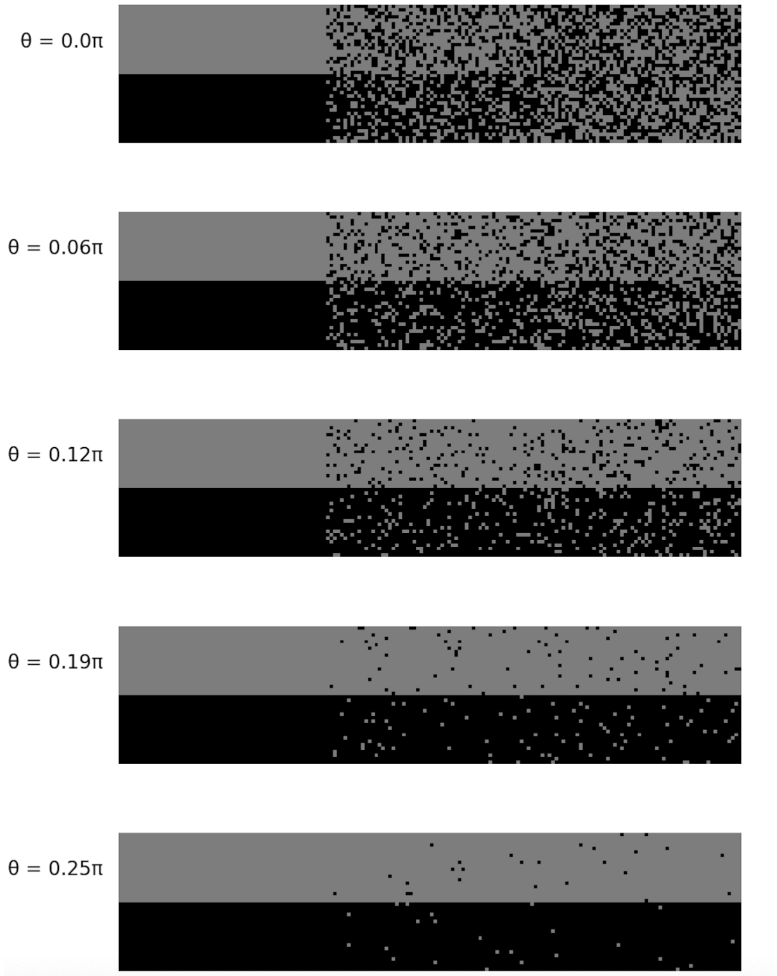

Figure 3. Comparison of the 3-state transportation and the 4-state transportation

The simulation shows that under the same level of interference ($\theta = \frac{\pi}{8}$), the three-state protocol has a higher discrimination accuracy compared to the four-state protocol. Hence, when a qutrit($0, 1, 2$) is more desired than a qubit($0, 1$), the 3-state protocol has a higher accuracy than the 4-state protocol, and it is optimal for superdense coding in such a scenario.

4. Variation of quantum teleportation with HSV model

4.1. Quantum teleportation of a single pixel with 3-level energy system

Although the 2-qubit 3-state protocol allows us to distinguish polluted state with a relatively high accuracy, it only allows us to distinguish 3 different states. In order to transfer more information, we consider a new protocol of quantum teleportation that allows to send 3 different complex numbers to the receiver.

Consider a three-level qutrit system:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle \tag{52}$$

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with $\alpha, \beta, \gamma \in \mathrm{C}$ and $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$. She also shares a maximally entangled qutrit pair with Bob:

$$|\Phi^+\rangle_{23} = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle). \tag{53}$$

Let qutrit 1 be the unkown state, qutrit 2 be Alice's part of the EPR pair and qutrit 3 be Bob's part of the EPR pair. The initial 3-qutrit system can be expressed as

$$|\Psi_{init}\rangle = |\psi\rangle_1 \otimes |\Phi^+\rangle_{23}. \tag{54}$$

If Alice wants to teleport this system to Bob without damaging it, she can apply the qutrit quantum teleportation protocol.

Similar to a classical qubit quantum teleportation protocol, it can be defined that bell states for a qutrit system. The Bell states are defined as

$$|\Phi_{mn}\rangle = \frac{1}{\sqrt{3}} \sum_{k=0}^{2} \omega^{kn} |k\rangle \otimes |k + m \, mod \, 3\rangle, \tag{55}$$

where $\omega = e^{2\pi i/3}$, and $m, n \in \{0, 1, 2\}$. These nine states form an orthonormal basis for the Hilbert space, hence the initial state can be expressed as a linear combination of these nine states.

$$|\Psi_{init}\rangle = \sum_{m=0}^{2} \sum_{n=0}^{2} \frac{1}{3} |\Phi_{mn}\rangle_{12} \otimes (X^m Z^n |\psi\rangle)_3, \tag{56}$$

Where X and Z are two operators given by:

$$X|j\rangle = |j + 1 \, mod \, 3\rangle, \quad Z|j\rangle = \omega^j |j\rangle. \tag{57}$$

When Alice performs a POVM on $|\Psi_{init}\rangle$, she obtains the measurement outcome (m, n) and collapses the qutrit 1 and qutrit 2 systems, leaving Bob's qutrit in the transformed state:

$$|\psi_{mn}\rangle = X^m Z^n |\psi\rangle. \tag{58}$$

After receiving (m, n) from Alice, Bob applies the inverse transformation to his half of the EPR pair:

$$(X^m Z^n)^\dagger = Z^{-n} X^{-m}, \tag{59}$$

recovering the unknown state that we want to deliver,

$$|\psi\rangle = Z^{-n} X^{-m} |\psi_{mn}\rangle. \tag{60}$$

This unique protocol allows us to transfer a quantum state to the receiver with a very high accuracy: even the sender cannot gain information from the quantum state. Upon receiving the state, the receiver can take measurements from the quantum state to measure the three coefficients: α, β, γ . However, this requires multiple copies of the quantum state to warrant a relatively high accuracy of the three coefficients. During each measurement, there is a chance of $|\alpha|^2, |\beta|^2$ and $|\gamma|^2$ to observe $|0\rangle, |1\rangle, |2\rangle$. Hence, statistically, the more measurements we conduct, the more accurate results we get for these three coefficients.

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With three real parameters $A=|\alpha|^2, B=|\beta|^2$ and $C=|\gamma|^2$, the HSV color model can be applied to "decode" these parameters into a specific color. H (Hue) represents the type of color; it is measured in degrees and describes the angle in the color wheel. It represents whether the color is red, green or blue etc. S (Saturation) is the color intensity, and it shows how "grey" the color is. V (Value) represents the brightness of the color, measured from black to the lightest version of the color.

However, since H is measured from 0 to 360 degrees, and S, V is measured from 0 to 1, we need to convert A, B, C t H, S, V, given by the following:

$$H = 360 \times A, S = B, V = C \tag{61}$$

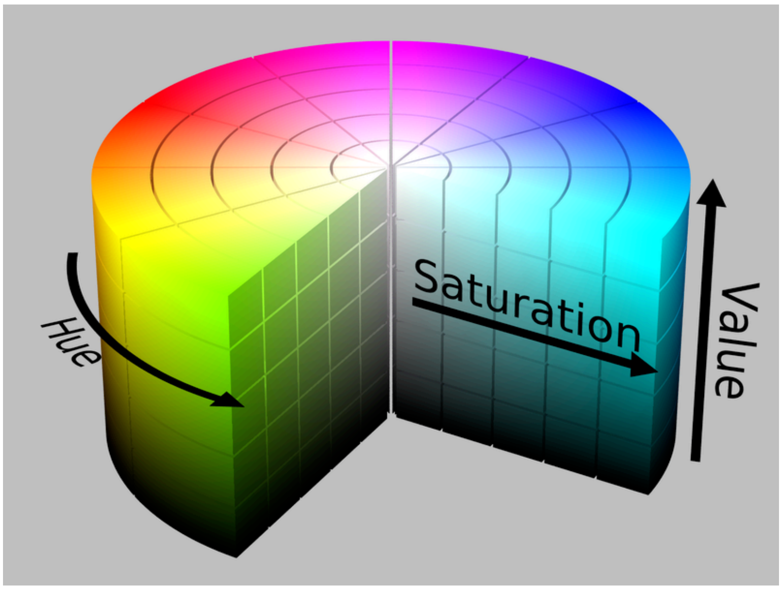

Figure 4. HSV model [12]

Combining the HSV model (Figure 4) and our protocol, it can teleport a very specific color to the receiver, as the three coefficients are quite arbitrary. This protocol allows us to potentially teleport images with colors in the future, through multiple 3-state quantum-teleportation on each pixel.

4.2. Integrated protocol of quantum teleportation and superdense coding

In this subsection, a protocol that integrates ideas from quantum teleportation and superdense coding to achieve a filter-like effect in image teleportation will be discussed.

As shown in Figure 5, consider a three-level qutrit system $|\psi\rangle_0$ that governs the color of a pixel using the HSV model; Charlie is introduced as the one who holds the "filter" quantum qubit c, and we make the qubit c to be entangled with $|\psi\rangle_1$ by a two-qubit unitary gate$U$on particle a and c. We obtain

$$|\psi\rangle_{1c} = U(|\psi\rangle_1 \otimes |c\rangle) = a|00\rangle_{1c} + b|11\rangle_{1c} + c|22\rangle_{1c} \tag{62}$$

Since the qubit c is now entangled with the qutrit system, it can apply unitary transformation of qubit c which can modify the entire state.

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$$\begin{aligned}(I\otimes V)|\psi\rangle_{1c} &= (I\otimes V)(a|00\rangle_{1c}+b|11\rangle_{1c}+c|22\rangle_{1c})\\ &= a|0,\alpha_0\rangle+b|1,\alpha_1\rangle+c|2,\alpha_2\rangle\\ &= a|0\rangle(h_0|0\rangle+h_1|1\rangle+h_2|2\rangle)\\ &\quad +b|1\rangle(s_0|0\rangle+s_1|1\rangle+s_2|2\rangle)\\ &\quad +c|2\rangle(v_0|0\rangle+v_1|1\rangle+v_2|2\rangle)\\ &= ah_0|00\rangle+bs_1|11\rangle+cv_2|22\rangle+|\phi\rangle\end{aligned} \tag{63}$$

Where h_i, s_i, v_i are arbitrary parameters which controls the Hue, Saturation and Value of the pixel, and it is determined by the unitary transformation V. ϕ denotes all the other states where for $|ij\rangle$, $i \neq j$.

Similarly to the protocol we defined before,

$$|\Psi_{init}\rangle = |\psi\rangle_{1c} \otimes |\Phi^+\rangle_{23}. \tag{64}$$

Furthermore, similar operations are conducted on $|\Psi_{init}\rangle$, where POVM is conducted on the initial state and retrieve the original state by applying the inverse operation given by

$$|\psi\rangle_{1c} = Z^{-n}X^{-m}|\psi_{mn}\rangle. \tag{65}$$

Since Bob now has the quantum state $|\psi\rangle_{1c}$, he can use a set of operators $\{\langle 00|, \langle 11|, \langle 22|\}$ to obtain the three parameters: ah_0, bs_1, cv_2.

In the context of image teleportation, Charlie can function as a "filter". If the user decides, for example, to change the brightness of the image, he can simply modify the value of v_2, so that the entire image(not a specific pixel) will be brighter or dimmer depending on the value of v_2 .

Furthermore, adding another perspective to this protocol also improves the security of the protocol. When Alice discovers that the state $|\psi\rangle_1$ is captured by attackers, she can inform Charlie so that Charlie can execute operations on the combined state $|\psi\rangle_{1c}$, so that the attacker will not be able to obtain the original state and get information from it.

In addition, Charlie can also be the "decoder" of the original state. Let the initial state $|\psi\rangle$ be encoded into $|\psi\rangle_{enc}$ by the following matrix, it can even be any arbitrary unitary transformation:

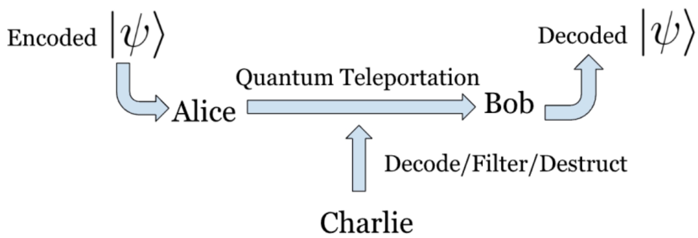

Figure 5. Integrated protocol of quantum teleportation and superdense coding

$$U_{enc} = F_3 = \frac{1}{\sqrt{3}}\begin{pmatrix}1 & 1 & 1\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega\end{pmatrix}, \quad \omega = e^{2\pi i/3}. \tag{66}$$

This maps the computational basis states $\{|0\rangle, |1\rangle, |2\rangle\}$ to an equal superposition basis. Since U_{enc} is unitary, its inverse is its Hermitian conjugate:

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$$U_{dec} := U_{enc}^{\dagger}. \tag{67}$$

Applying U_{dec} recovers the original state:

$$|\psi\rangle = U_{dec}|\psi_{enc}\rangle.$$

In this highly-theoretical protocol, neither Alice nor Charilie have access to $|\psi\rangle$. Alice only holds the encoded state and she doesn't know how to encode it. Charlie only has the method to decode the state but he will never have actual access to qubit 1. Both Alice and Charlie are essential during the teleportation process. Only if Alice and Charlie collaborate, the receiver will be able to receive the correct $|\psi\rangle$ that's not encoded, and gain information from it (color of the pixel etc).

5. Conclusion

In this paper, two distinct "3-state" protocols that can be potentially applied for image teleportation have been discussed. For the first protocol, super-dense coding with three different Bell states under a noisy environment was considered. The Helstrom theorem was applied for the analysis of the accuracy of transportation. The three Bell states in the protocol each represent a color: black, white, and gray. One hundred percent accuracy for the color white and a relatively high accuracy for the other two colors were achieved, compared to the classical 4-state superdense coding protocol. For the second protocol, a new protocol for quantum teleportation, which utilizes a 3-level energy system, has been discussed. Furthermore, the protocol was connected to the HSV color model, which allows the 3-level system to be "decoded" to a specific color. In conclusion, both of these protocols can be potentially applied for image teleportation in the future, and it has been proven that either they have a higher accuracy, or they can be related to a specific existing technique in image teleportation. Further application of these protocols is to test them under experimental conditions so that they can be used for scenarios where image teleportation requires very high confidentiality.

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