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The Chromatic Index of Asymmetric Pendant Helm Graphs —A Constructive Proof of the Class 1 Property for a Generalized Helm Family

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Abstract. The classical helm graph is obtained from a wheel graph by attaching a single pendant edge to each vertex on its outer cycle. While the chromatic index of the classical helm graph has been determined previously, that for nonuniform pendant extensions remains unexplored. In this paper, we introduce the asymmetric pendant helm graph, a natural generalization in which each outer vertex may carry an arbitrary number of pendant edges. We prove that every asymmetric pendant helm graph is Class 1, with the chromatic index equal to the maximum degree. The proof is constructive and yields an optimal linear-time edge coloring algorithm. As special cases, the result recovers known conclusions for classical helm graphs, wheel graphs, and uniform multi-pendant helm graphs. We further validate the proposed construction computationally using ten representative graph instances.

Keywords: edge coloring, chromatic index, helm graph, Class 1 graph, constructive proof

1. Introduction

Edge-coloring is a fundamental topic in graph theory whose origins trace to the work of Petersen [1] and Kőnig [2]. Vizing's theorem [3, 4] classifies all simple graphs into two categories: Class 1 graphs satisfy $\chi^{'}(G)=\Delta(G)$, while Class 2 graphs satisfy $\chi^{'}(G)=\Delta(G)+1$. Determining the exact class of a given graph is NP-complete in general [5], thus driving research to identify infinite families of Class 1 graphs. The classical helm graph H_n—formed by attaching one pendant edge to each outer vertex of a wheel W_n—is known to be Class 1 [6], and helm-related graphs have attracted sustained attention in edge coloring research [7-9]. However, all prior work focuses on symmetric constructions in which outer vertices carry identical numbers of pendant edges. The natural question of what happens when pendant counts vary across different outer vertices remains unexamined.

In this paper, we introduce the *asymmetric pendant helm graph* $H_n(p)$, where each outer vertex u_i receives an arbitrary number p_i of pendant edges. We ask whether every graph in this generalized family remains Class 1 and whether the chromatic index can be expressed in closed form.

We answer affirmatively through a constructive proof that yields an explicit optimal edge coloring using exactly $\Delta(H_n(p))$ colors.

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This result extends the Class 1 characterization from symmetric to fully asymmetric pendant distributions, demonstrating that the Class 1 property is remarkably robust under arbitrary nonuniform extensions. The construction also provides a practical linear-time coloring algorithm, and as special cases, it recovers known results for classical helm graphs, wheel graphs, and uniform multi-pendant helm graphs.

2. Asymmetric pendant helm graphs

2.1. Preliminaries

All graphs in this paper are finite, undirected, and simple. We follow standard notation [10-12].

For a graph G, $V(G)$ and $E(G)$ stand for its vertex and edge sets, and $d_G(v)$ represents the degree of a vertex v. The maximum degree is $\Delta(G)=\max_{v\in V(G)} d_G(v)$. Definition 2.1 (Proper edge-coloring and chromatic index). *A proper* k*-edge coloring of* G *is a mapping* $\varphi:E(G)\to\{1,...,k\}$ *satisfying* $\varphi(e)\neq\varphi(e')$ *whenever* e *and* e' *share a common endpoint. The chromatic index* $\chi'(G)$ *is the minimal integer* k *admitting such a coloring.*

Definition 2.2 (Class 1 and Class 2). G is Class 1 if $\chi'(G)=\Delta(G)$ and Class 2 if $\chi'(G)=\Delta(G)+1$.

The wheel graph W_n ($n\geq 3$) has vertex set $\{c\}\cup\{u_1,...,u_n\}$, where c is the center connected to all outer vertices, and the outer vertices form a cycle $u_1u_2\cdots u_nu_1$. It is known that $\chi'(W_n)=\Delta(W_n)=n$; consequently, W_n falls into Class 1 [6, 10].

The classical helm graph H_n adds one pendant vertex v_i adjacent to each u_i. Its degree sequence is $d(c)=n$, $d(u_i)=4$, $d(v_i)=1$, giving $\Delta(H_n)=\max\{n,4\}$. As noted, $\chi'(H_n)=\max\{n,4\}$ [6].

Given $n\geq 1$ and any integer r, we define the modular color index

$$\langle r\rangle_n=1+((r-1) \bmod n) \tag{1}$$

which satisfies $\langle r\rangle_n\in\{1,...,n\}$ for all r. This operation assigns consecutive integers to a circular color pattern over the outer cycle.

2.2. Definition and basic properties

Definition 2.3 (Asymmetric pendant helm graph). *Let* $n\geq 3$ *and* $p=(p_1,...,p_n)\in Z^n_{\geq 0}$. *The asymmetric pendant helm graph* $H_n(p)$ *is constructed from* W_n *by attaching* p_i *pendant vertices to each outer vertex* u_i.

The vertex and edge sets are formally specified as:

$$V(H_n(p))\& = \{c\} \cup \{u_1, \ldots, u_n\} \cup \{v_{i,j} : 1 \leq i \leq n,\ 1 \leq j \leq p_i\}, \tag{2}$$

$$E(H_n(p))\& = E_s \cup E_c \cup E_p,$$

where $E_s=\{cu_i:1\leq i\leq n\}$ collects the spoke edges, $E_c=\{u_iu_{i+1}:1\leq i\leq n\}$ (indices modulo n) contains the cycle edges, and $E_p=\{u_iv_{i,j}:1\leq i\leq n, 1\leq j\leq p_i\}$ denotes the pendant edges.

Proposition 2.4 (Degree sequence).

In $H_n(p)$: $d(c)=n$, $d(u_i)=p_i+3$, $d(v_{i,j})=1$.

Let $p_{max}=\max_i p_i$. The maximum degree follows immediately:

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$$\Delta(H_n(p))=\max\{n,\ p_{max}+3\} \tag{3}$$

The total number of edges is

$$|E(H_n(p))|=2n+\sum_{i=1}^{n} p_i \tag{4}$$

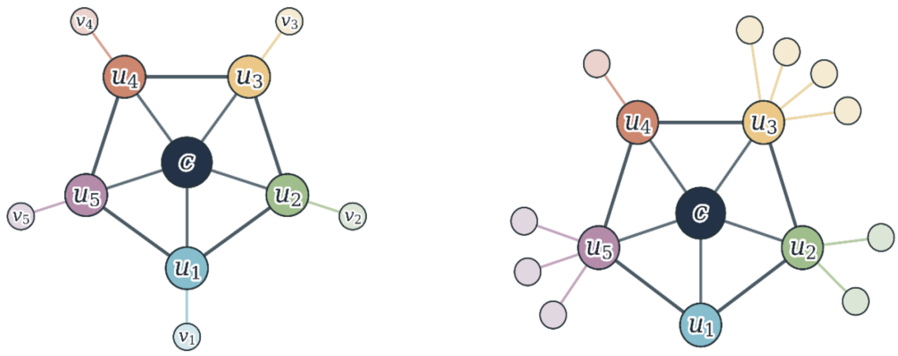

Figure 1. Left: Classical helm graph H_5 (all p_i=1). Right: Asymmetric pendant helm graph $H_5(0,2,4,1,3)$

Red circular markers represent the central c, blue circles mark outer vertices u_i, and green squares stand for pendant vertices as shown in figure 1.

3. Exact chromatic index

Set

$$K=\Delta(H_n(p))=\max\{n,\ p_{max}+3\} \tag{5}$$

The color set is $C=\{1,...,K\}$.

Lemma 3.1 (Lower bound). $\chi'(H_n(p))\geq K$.

Proof. All edges incident to a vertex of maximum degree carry mutually distinct colors in any proper edge-coloring, which immediately implies $\chi'(G)\geq\Delta(G)$ for any graph G.

3.1. Coloring the spoke and cycle edges

We first color the 2n edges in $E_s\cup E_c$ using only colors $1,...,n\subseteq C$.

Lemma 3.2 (Spoke and cycle edge coloring). *The assignment* $\varphi(cu_i)=i$, $\varphi(u_iu_{i+1})=\langle i+2\rangle_n(1\leq i\leq n)$ *yields a proper edge coloring of* $E_s\cup E_c$. *At each outer vertex* u_i, *the three incident edges from* $E_s\cup E_c$ *carry the distinct colors* $\langle i\rangle_n$, $\langle i+1\rangle_n$, *and* $\langle i+2\rangle_n$.

Proof. The spoke edge cu_i receives color $i=\langle i\rangle_n$. The cycle edge $u_{i-1}u_i$ receives $\langle (i-1)+2\rangle_n=\langle i+1\rangle_n$, and u_iu_{i+1} receives $\langle i+2\rangle_n$. Since $n\geq 3$, no two of $\{i,i+1,i+2\}$ differ by a multiple of n; hence the three values $\langle i\rangle_n$, $\langle i+1\rangle_n$,$\langle i+2\rangle_n$ are all distinct. At the center c, the n spoke edges carry the n distinct colors $1,...,n$.

Thus after coloring $E_s\cup E_c$, at each u_i exactly three colors from $\{1,...,n\}$ are already used.

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3.2. Coloring the pendant edges

Lemma 3.3 (Sufficient available colors). *For every* i, $|C\setminus\Phi_i|\geq p_i$, *where* Φ_i *is the set of three colors used on* $E_s\cup E_c$ *at* u_i.

Proof. Φ_i has size exactly 3 by Lemma 3.2. Thus $|C\setminus\Phi_i|=K-3$. Since $K\geq p_{max}+3\geq p_i+3$, we have $K-3\geq p_i$.

Lemma 3.3 confirms that at least p_i residual unused colors from C remain available at each outer-cycle vertex u_i. We allocate these injectively to the p_i pendant edges incident to u_i.

3.3. Main theorem

Theorem 3.4 (Chromatic index of asymmetric pendant helm graphs). *For every* $n\geq 3$ *and every* $p\in Z^n_{\geq 0}$, $\chi'(H_n(p))=\Delta(H_n(p))=\max\{n, p_{max}+3\}$. *In particular, every* $H_n(p)$ *is a Class1 graph.*

Proof. The lower bound is Lemma 3.1. For the upper bound, the construction in Lemmas 3.2-3.3 yields a proper edge coloring using at most K colors. Every pendant vertex has degree 1, so no conflicts arise from pendant edge assignments. Thus $\chi'(H_n(p))\leq K$, and the equality follows.

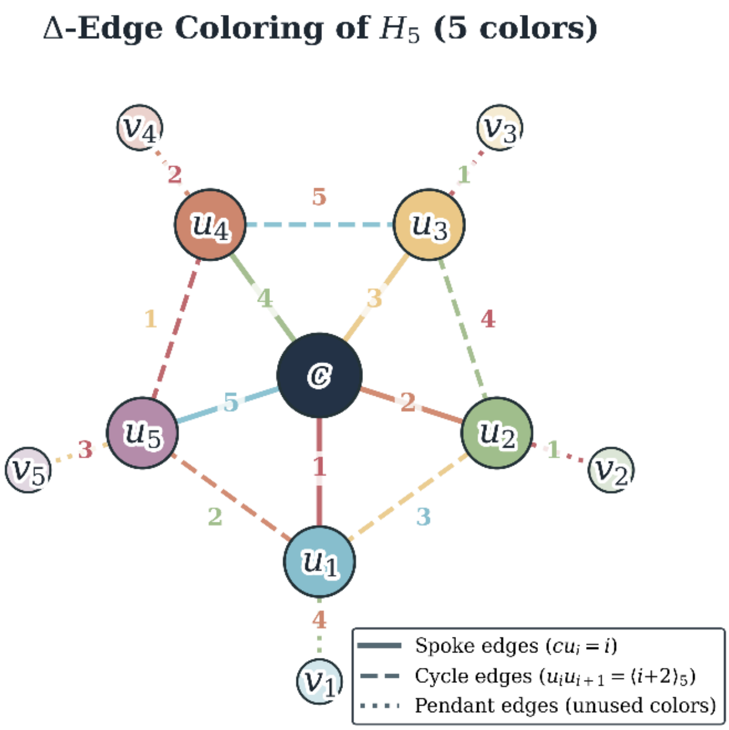

Figure 2. Edge coloring of H_5 using Δ=5 colors. Solid lines: Spoke edges colored by index. Dashed lines: Cycle edges colored by modular rule. Dotted lines: Pendant edges filled with unused local colors

4. Optimal edge coloring algorithm

The constructive proof translates directly into Algorithm 1.

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Algorithm 1 OptimalEdgeColoring(n,p)

Require: n≥3,vector p=(p_1,..., p_n)of nonnegative integers
Ensure: A proper K-edge coloring φ of H_n(p)with K=max{n,Pmax+3} 1:Pmax←max{p1,...,Pn}
2: K←max{n,P_{max}+3}
3: for i=1 to n do
4: (cui)←i Spoke edges
5: end for
6: for i=1 to n do
7: φ(uiUi+1)←1+((i+1)mod n) Cycle edges(indices mod n)
8: end for
9: for i=1 to n do
10: i←{4(cu₂),φ(ui-1ui),φ(u₂ui+1)}
11: Ai←{1,...,K}\Φi D Available colors at ui
12: for j=1 to pi do
13: Choose α∈Ai;φ(uiVi,j)←α;Ai←Ai\{a}
14: end for
15: end for
16: return φ

Proposition 4.1 (Correctness). Algorithm 1 returns a proper K-edge coloring of $H_n(p)$.

Proof. Lines 3-5 implement Lemma 3.2, ensuring all spoke and cycle edges are properly colored. Lines 6-11 assign pendant edges from the available colors at each u_i; Lemma 3.3 guarantees $|A_i| \geq p_i$. Pendant vertices have degree 1, so no additional conflicts arise.

Proposition 4.2 (Complexity). *Algorithm1 runs in* $O(|E(H_n(p))|)$ *time and uses exactly* $\Delta(H_n(p))$ *colors.*

Proof. Computing p_{max} takes $O(n)$. The spoke and cycle loops each take $O(n)$. For each u_i, computing A_i takes $O(K)$ time and assigning p_i pendant edges takes $O(p_i)$. Summing over all vertices and noting that $K \leq n+p_{max}+3$, the total time is $O(nK+\sum p_i)=O(|E|)$. The algorithm uses exactly $K=\Delta$ colors with no recoloring.

5. Corollaries

Corollary 5.1 (Classical helm graph). $\chi'(H_n)=\max\{n,4\}$. *Thus* H_3 *requires* 4 *colors and* H_n *(* $n\geq4$ *) requires* n *colors.*

Corollary 5.2 (Wheel graph). for all integers $n\geq3$, $\chi'(W_n)=n$. This recovers the known result as the $p_i\equiv0$ case. Corollary 5.3 (Uniform multi-pendant helm graph). Suppose all $p_i\equiv t$, then $\chi'(H_n(t))=\max\{n,t+3\}$. Corollary 5.4 (Asymmetric dominance). If $p_{max}\geq n-3$, then $\chi'(H_n(p))=p_{max}+3$. The chromatic index is determined by the single vertex with the most pendant edges.

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6. Examples and computational verification

6.1. Illustrative example: H_5

For the classical helm graph H_5, we have $K=\max\{5,4\}=5$. Table 1 shows the constructed coloring of spoke and cycle edges. Pendant edges u_iv_i are then assigned arbitrary unused colors at each u_i (e.g., colors 4,1,1,2,3, respectively).

Table 1. Spoke and cycle edge coloring of H_5

Spoke edge	Color	Cycle edge	Color
cu_1	1	u_1u_2	$\langle 3\rangle_5=3$
cu_2	2	u_2u_3	$\langle 4\rangle_5=4$
cu_3	3	u_3u_4	$\langle 5\rangle_5=5$
cu_4	4	u_4u_5	$\langle 6\rangle_5=1$
cu_5	5	u_5u_1	$\langle 7\rangle_5=2$

6.2. Computational verification

We coded Algorithm 1 in Python and validated the correctness of its produced edge-colorings across ten diverse instances spanning classical helm graphs ($n=3,4,5,6$), nonuniform cases, wheel graphs, large uniform pendants, and alternating patterns. In every instance, the algorithm produced a proper edge coloring using exactly Δ colors, confirming the theoretical result. The verification code and results are summarized in Table 2.

Table 2. Computational verification of the constructive coloring algorithm. All constructed colorings use exactly $\Delta(H_n(p))$ colors and pass adjacency validation

n	p	Δ	Colors used	Valid?
3	(1, 1, 1)	4	4	yes
4	(1, 1, 1, 1)	4	4	yes
5	(1, 1, 1, 1, 1)	5	5	yes
6	(1, 1, 1, 1, 1, 1)	6	6	yes
5	(0, 2, 4, 1, 3)	7	7	yes
8	(2, 0, 5, 1, 3, 4, 2, 1)	8	8	yes
7	(3, 5, 0, 2, 4, 1, 2)	8	8	yes
6	(0, 0, 0, 0, 0, 0)	6	6	yes
10	(5, 5, 5, 5, 5, 5, 5, 5, 5, 5)	10	10	yes
8	(0, 1, 0, 2, 0, 3, 0, 4)	8	8	yes

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7. Discussion

7.1. Why the parity intuition fails

One intuitively appealing yet false conjecture claims that the parity of n determines whether H_n is Class 1 or Class 2, mirroring the parity-based classification rule for complete graphs K_n. Our explicit coloring of H_5 with 5 colors directly refutes this intuition. The reason is structural: the pendant edges at different outer vertices are mutually nonadjacent and can freely reuse colors, so no global parity obstruction emerges. The three colors used on spoke and cycle edges at u_i are always three consecutive residues modulo n, leaving $K-3 \geq p_i$ colors for the local pendant edges regardless of n's parity.

7.2. Stability of the Class 1 property

A core takeaway emerging from our main theorem concerns the striking robustness of the Class 1 property under arbitrary pendant vertex attachments to outer-cycle vertices. Starting from a wheel graph (already Class 1), adding arbitrary pendant edges to outer vertices increases both local demand and the global color budget in a balanced way: the center's degree n provides a floor, and the most pendant-heavy vertex drives the ceiling $p_{max}+3$. Since pendant edges incident to distinct outer-cycle vertices operate under fully independent color assignment rules, no inter-vertex color collisions can push the graph into Class 2.

This explains the contrast with closed helm graphs [7], which do exhibit more complex behavior: closing the pendant cycle creates mutual adjacency among pendant edges, reintroducing global constraints.

8. Conclusion

We define the asymmetric pendant helm graph $H_n(p)$—a natural generalization of the classical helm graph in which each outer vertices may carry different numbers of pendant edges—and prove that all graphs in this family belong to Class 1, satisfying $\chi'(H_n(p))=\max\{n, p_{max}+3\}$. The proof is constructive, yielding an optimal linear-time coloring algorithm. Computational verification on ten diverse instances confirmed the theoretical result. This work opens up multiple promising avenues for subsequent graph-theoretic investigation. The List Edge Coloring Conjecture asserts $\chi'_1(G)=\chi'(G)$ for all graphs G; determining whether $\chi'_1(H_n(p))=\Delta(H_n(p))$ holds for the asymmetric family would test this conjecture in a structured but nonuniform setting. A star edge coloring forbids bichromatic paths of length 3, and Lei and Shi's survey [9] identifies helm and wheel graphs as open cases; determining the star chromatic index of $H_n(p)$ is a natural extension where the asymmetry and parity of n may play a role. Adding a cycle among pendant vertices, analogous to the closed helm graph [7], would reintroduce cross-vertex constraints; we conjecture that parity-driven Class 2 behavior emerges in this closed variant. Finally, strong edge coloring, which requires edges at distance at most 2 to receive distinct colors, couples pendant edges at different u_i and may lead to classification outcomes governed by the graph's structural parameters.

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