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Application of Residue Theorem in Trigonometric Function Integration

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Abstract. The calculation of definite integrals is typically complex and might have an error, which demands a more generalized and convenient method. The traditional techniques like universal substitution and integration by parts tend to give complicated integrands and time-consuming calculations. These definite integral problems are then converted by the residue theorem into straightforward summation problems of definite residues via contour integration on the complex plane to significantly decrease number of steps in calculations. By using the Laurent series to expand a complex function with a simple pole in denominator and take the limit as approaches the pole, a general formula to calculate residue of simple poles is given. In terms of a singularity with order at the pole, taking derivative of the series leads to the residue. The overall formula can be directly used on definite integral with trig in denominator with the appropriate choice of root of the function inside the unit circle. Using the transformation between the functions of sine and cosine, the limit can be simplified to obtain the residue. These conversions allow the general formulae of calculating a residue and through the residue theorem the resultant integral is simply obtained.

Keywords: definite integral, complex trigonometric function, residue theorem

1. Introduction

The complex function theory is a significant subline of mathematics because the computation of definite integrals using real analysis is usually not easy and efficient. One of the most useful tools of a complex analysis will be the residue theorem, which relates the integrals of complex functions on a closed contour and the residues of the singularities within the contour, offering a compact yet powerful way of calculating definite integrals [1]. It has been primarily used in the calculation of trigonometric integrals [2]. Trigonometric integral is often used in the study of a variety of disciplines. It has found extensive use in engineering in the study of resonance circuit design and antenna design. In physics, it may occur in studies of topological phase transitions [3]. It is, however, sometimes not handled by existing methods like Fourier series and Weier strass substitution because of a special property of its rational fractions. Hence, the scholars shift their attention to the transformation of the real trigonometric integrals into the complex variable functions in numerical derivations.

The practical usefulness of the residue theory may have been more in the resolution of practical problems. An example is that both Fourier analysis and wave propagation models require a highly effective and accurate mathematical tool to compute the harmonic component of the calculations and integrals respectively. As long as the residue theorem has the ability to handle these integrals in a systematic manner and provide mature algorithmic frameworks, the research work with its applications has practical importance. Moreover, the training of trigonometric integrals is commonly important but challenging in complex variable courses in majors in both math and engineering. Using an intellectual bridge between real and complex integrals, studies on the application of the residue theorem can be used to develop students' innovative capacity and model thinking and enhance the potential of the theory of complex functions in developing further application in more fields.

According to Zhong Yuquan, residue theorem is a tool that is absolutely necessary in the calculation of contour integral [4]. The residue theorem has since been extensively used in computing real integrals since the original invention by Cauchy of his underlying theories. In particular, it is able to eliminate typical errors when computing generalized integrals of trigonometric functions [5]. The classical analogous lemma that was heavily relied upon in the past to obtain infinite integrations of trigonometric functions has limited the scope of use of the contour integration method to a large extent [6]. Comparatively, contour integration via the residue theorem offers a high level of convenience once the right integrand and the right contour are chosen [7], which may considerably ease the computations.

This work is intended to help readers detail a few examples of various types of trig function integrals to show both the merits and demerits of current methods of applying the residue theorem. The remainder of the paper is structured in the following way. Sec. 2 first examines the derivation of residue and the residue theorem, and then briefly discusses how to compute residue under two circumstances of the pole using the Laurent expansion. Sec. 3 gives an example of five trig functions to demonstrate how the residue theorem can be used to solve real world problems involving trigonometry by contour integration.

2. Methods

2.1. The residue theorem

The complex theory has undergone numerous studies in which many scholars have made it standardized and more powerful in its applications. Integration of rational fractions containing trigonometric functions Zhang Qifeng has addressed the integration of such fractions in his 2026 paper [7]; Yang Shijie has written about how to use the residue theory in calculating an infinite series sum involving the hyperbolic cosine function [8]; Ahlfors has systematically described the standard choice of contours used in solving trigonometric integrals with residues in his The theorem was applied by Gordan to compute improper integrals of sine and cosine functions [9]. Nevertheless, the research on the uses of the residue theorem in trigonometric functions is yet to be complete. To begin with, there is no discussion on parametric trig functions. Secondly, there are no effective ways to notice the kind of singularity, particularly an easily misunderstood one such as a removable singularity referred to as a pole [10].

If the Laurent expansion of a function $f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n$ is integrated into a simple closed contour C around the isolated singularity z_0 , the paper obtains

$$a_n\int(z-z_0)^n dz=a_n\frac{(z-z_0)^{n+1}}{n+1}=0 \tag{1}$$

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where $n \neq -1$. Then, after plugging in $n = -1$, this becomes

$$a_{-1} \int (z - z_0)^{-1} dz = a_{-1} \int \frac{ire^{i\theta}}{re^{i\theta}} d\theta = 2\pi i a_{-1} \tag{2}$$

By expressing the left-hand side of equation (2) in the form of the left-hand side of equation (1), which gives $\int f(z)dz$, and dividing this and the right-hand side of (2) by 2 πi , the paper has $a_{-1} = \frac{1}{2\pi i} \int f(z)dz$. Define a_{-1} to be the residue of $f(z_0)$, one then can denote it as

$$Res(f,\ z_0) = \frac{1}{2\pi i} \oint_C f(z)dz \tag{3}$$

It is obvious that if $f(z)$ is analytic, the residue of it will be zero by Cauchy's integral theorem. This leads to the derivation of an essential theorem in evaluating definite integrals in complex analysis. Assume a finite number of isolated singularities $z_{0,}\ z_1,\ z_2 \cdots$ in a simple closed contour C. By deforming C to exclude all the singularities, Cauchy's integral theorem gives

$$\oint_C f(z)dz + \oint_{C_0} f(z)dz + \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \cdots = 0 \tag{4}$$

where $C_0,\ C_1,\ C_2 \cdots$ are the circular integrals around each pole. According to equation (2), the circular integral around a singularity is

$$\oint_{C_i} f(z)dz = 2\pi i a_{-1,\ z_i} \tag{5}$$

Combining Eqs. (4) and (5), the residue theorem is given

$$\oint_C f(z)dz = 2\pi i \left(a_{-1,z_0} + a_{-1,z_1} + a_{-1,z_2} + \cdots\right) \tag{6}$$

The usefulness of this theorem will show up later, as the residues of $f(z)$ can be calculated algebraically to obtain a contour integral which is comparatively easy to other methods. Since calculating definite integrals by using methods in real analysis is usually inefficient, by transforming real variable integrals into complex ones, and selecting an appropriate complex-plane contour, problems can be greatly simplified.

2.2. Calculation of residues

Recalling the process of applying the Laurent series to obtain residue, there are more general methods of calculating it based on specific situations. The article will show how to manipulate them to easily and efficiently calculate residues.

The first situation is when the singularity is a simple pole z_0 . Thus, the Laurent series of it is

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$$f(z) = \sum_{n=0}^{\infty} A_n(z-z_0)^n + \frac{A_{-1}}{z-z_0} = \frac{g(z)}{(z-z_0)} \tag{7}$$

where $g(z) = \sum_{n=0}^{\infty} A_n(z-z_0)^{n+1} + A_{-1}$. Then, the residue at z_0 can be expressed as

$$A_{-1} = \lim_{z\to z_0}(z-z_0)f(z) = g(z_0\Big) \tag{8}$$

Take the function $f(z) = \frac{z^3}{(z-4)}$ to correspond to the form of $f(z) = \frac{g(z)}{z-z_0}$, where $z_0 = 4$ and $g(z\ )=\ z^3$. Then, this paper calculates the residue by taking the limit as z approaches 4 of $(z-4)\frac{z^3}{z-4} = 4^3 = 64$. This paper can verify the answer by using the Laurent expansion around $z_0 = 4$, which is

$$f(z) = \frac{(z-4+4)^3}{z-4} = (z-4)^2 + 12(z-4) + 48 + \frac{64}{z-4} \tag{9}$$

The coefficient of $\frac{1}{z-4}$ is 64, corresponding to A_{-1} which is the residue gotten at first.

The second situation is when the singularity has an order m at the pole. Using the Laurent expansion,

$$f(z) = \sum_{n=0}^{\infty} A_n\big(z-z_{0)}^n + \frac{A_{-1}}{(z-z_0)} + \frac{A_{-2}}{(z-z_0)^2} + \cdots + \frac{A_{-m}}{(z-z_0)^m} = \frac{g(z)}{(z-z_0)^m} \tag{10}$$

Then

$$g(z) = \sum_{n=0}^{\infty} A_n(z-z_0)^{n+m} + A_{-1}(z-z_0)^{m-1} + A_{-2}(z-z_0)^{m-2} + \cdots + A_{-m} \tag{11}$$

Here it can be seen that

$$(z-z_0)^m f(z) = \sum_{n=0}^{\infty} A_n(z-z_0)^{n+m} + A_{-1}(z-z_0)^{m-1} + A_{-2}(z-z_0)^{m-2} + \cdots + A_{-m} \tag{12}$$

The author then takes the $(m-1)$ th derivative of both sides of equation (12), and gets

$$\frac{d^{m-1}[(z-z_0)^m f(z)]}{dz^{m-1}} = \sum_{n=0}^{\infty}(n+m)(n+m-1)\cdots(n+1)A_n(z-z_0)^{n+1} + (m-1)!A_{-1} \tag{13}$$

Therefore, the residue at z_0 can be expressed as

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$$Res\,(f, z_0) = \frac{1}{(m-1)!}\left[\frac{d^{m-1}}{dz^{m-1}}\, g\,(z)\right]_{z=z_0} \quad (14)$$

Here is an example. Let $f\,(z) = \frac{1}{(z-3)^2(z+3)^2}$, the only pole is $z_0 = 3$ of order 2. By using equation (14), it can be shown that

$$Res\,(f, z=3) = \frac{1}{1!}\frac{d}{dz}\left[(z-3)^2\frac{1}{(z-3)^2(z+3)^2}\right]_{z=3} = \frac{d}{dz}\left[(z+3)^{-2}\right]_{z=3} = -\frac{1}{72} \quad (15)$$

3. Applications

In what follows, the author shall take five examples to demonstrate the power of Residue theorem in calculating the trigonometric definite integrals. First, one can look at

$$I = \int_0^{2\pi} f(\sin\theta, \cos\theta\Big)\, d\theta \quad (16)$$

given that f is finite for all values of θ and rational of $\sin\theta$ and $\cos\theta$ since it should be single-valued. It is known $z = e^{i\theta},\ dz = ie^{i\theta}d\theta$. Therefore, $d\theta = -i\frac{dz}{z},\ sin\theta = \frac{z-z^{-1}}{2i},\ cos\theta = \frac{z+z^{-1}}{2}$. The author can then express the integral as

$$I = -i\oint f\left(\frac{z-z^{-1}}{2i}, \frac{z+z^{-1}}{2}\right)\frac{dz}{z} \quad (17)$$

by the contour integration. According to the residue theorem,

$$I = (-i)2\pi i \sum sum\ of\ the\ residues\ within\ the\ unit\ circle \quad (18)$$

Knowing this general form of contour integral, one can now use it to evaluate many trigonometric definite integrals. The first example is

$$I = \int_0^{2\pi}\frac{d\theta}{1+\varepsilon\cos\theta} \quad (19)$$

where $|\varepsilon| < 1$. By Eq. (17), this becomes

$$I = -i\oint\frac{dz}{z[1+(\varepsilon/2)(z+z^{-1})]} = -i\frac{2}{\epsilon}\oint\frac{dz}{z^2+(2/\varepsilon)z+1} \quad (20)$$

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The two roots of the denominator are $z_- = -\frac{1}{\varepsilon} - \frac{1}{\varepsilon}\sqrt{1-\varepsilon^2}$ and $z_+ = -\frac{1}{\varepsilon} + \frac{1}{\varepsilon}\sqrt{1-\varepsilon^2}$, but only z_+ is within the unit circle. Substituting it into equation (18), $I = -i\frac{2}{\varepsilon} * 2\pi i \frac{1}{2z+2/\varepsilon}\Big|_{z=-1/\varepsilon+\left(1/\varepsilon\right)\sqrt{1-\varepsilon^2}}$. Therefore, one gets

$$\int_0^{2\pi} \frac{d\theta}{1+\varepsilon\cos\theta} = \frac{2\pi}{\sqrt{1-\varepsilon^2}} \tag{21}$$

where $|\varepsilon| < 1$.

The second example is to evaluate $\oint_{|z|=2} \frac{\tan z}{z} dz$. Since the singularity at $z = 0$ is removable, the only two poles are at $\frac{\pi}{2}$ and $-\frac{\pi}{2}$. According to the residue theorem, the integral is equal to $2\pi i \left[Res\left(\frac{\tan z}{z}, \frac{\pi}{2}\right) + Res\left(\frac{\tan z}{z}, -\frac{\pi}{2}\right)\right]$. Noting that $\frac{\pi}{2}$ is a simple pole and $\cos z = -\sin\left(z - \frac{\pi}{2}\right)$, this article gets

$$Res\left(\frac{\tan z}{z}, \frac{\pi}{2}\right) = \lim_{z\to\frac{\pi}{2}} \frac{\left(z-\frac{\pi}{2}\right)\tan z}{z} = \lim_{z\to\frac{\pi}{2}} \frac{\left(z-\frac{\pi}{2}\right)\sin z}{z\left[-\left(z-\frac{\pi}{2}\right)+\frac{\left(z-\frac{\pi}{2}\right)^3}{6}+\cdots\right]} = -\frac{2}{\pi} \tag{22}$$

Since $\tan z/z$ is even, it can be assumed that $Res\left(\frac{\tan z}{z}, -\frac{\pi}{2}\right) = \frac{2}{\pi}$. Therefore, one has the conclusion $\oint_{|z|=2} \frac{\tan z}{z} \mathrm{dz} = 0$.

Next, evaluate the integral $I = \int_0^{2\pi} \frac{\cos 2\theta}{5-4\cos\theta} d\theta$. Since $z = e^{i\theta}$ for any z lying in the unit circle C where $0 \le \theta \le 2\pi$, $z^2 = \cos 2\theta + i\sin 2\theta,\ z^{-2} = \cos 2\theta - i\sin 2\theta$, and $\cos 2\theta = \frac{z^2+z^{-2}}{2}$ and $\sin 2\theta = \frac{z^2-z^{-2}}{2}$. Then, the integrand becomes

$$f(z) = \frac{\frac{1}{2}\left(z^2+z^{-2}\right)}{iz[5-2(z+z^{-1})]} = \frac{i\left(z^4+1\right)}{2z^2(z-2)(2z-1)} \tag{23}$$

The singularities of $f(z)$ lying inside the unit circle are the poles at $z_1 = 0$ of order 2 and $z_2 = \frac{1}{2}$. The author can now calculate the residues at these poles.

$$Res\,(f(z),\ 0) = \lim_{z\to 0}\frac{d}{dz} z^2 f(z) = \lim_{z\to 0}\frac{d}{dz}\left(\frac{i\left(z^4+1\right)}{2(2z^2-5z+2)}\right) = \frac{5i}{8} \tag{24}$$

And

$$Res\left(f(z),\ \frac{1}{2}\right) = \lim_{z\to\frac{1}{2}}\left(z - \frac{1}{2}\right)f(z) = \lim_{z\to\frac{1}{2}}\left(\frac{i\left(z^4+1\right)}{4z^2(z-2)}\right) = \frac{-17i}{24} \tag{25}$$

Then, according to the residue theorem,

$$\int_0^{2\pi} \frac{\cos 2\theta d\theta}{5-4\cos\theta} = 2\pi i \left(\frac{5i}{8} - \frac{17i}{24}\right) = \frac{\pi}{6} \tag{26}$$

Lastly, evaluate the integral $\int \frac{\sin 2z}{z^2} dz$. By writing the series representation of $\sin 2z$,

$$\sin 2z = 2z - \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} - \frac{(2z)^7}{7!} + \cdots \tag{27}$$

and entering it into the integrand, the paper gets

$$\frac{2z - \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} - \frac{(2z)^7}{7!} + \cdots}{z^2} = \frac{2}{z} - \frac{4z}{3} + \frac{4z^3}{15} - \frac{8z^5}{315} + \cdots, \tag{28}$$

which is the Laurent series of the function near $z = 0$. It is obvious that the numerator and denominator both equal to zero at $z = 0$. However, since $\sin 2z = 0$ when $2z = n\pi$ where $n = \pm 1, \pm 2, \cdots$, and z^2 is not equal to 0 at these values, the author can just focus on the function's behavior at $z = 0$. From equation (31), it is shown that the coefficient of the term with z of power negative one is 2. Therefore, $Res\ f(0) = 2$ and $I = \int \frac{\sin 2z}{z^2} = 2\pi i \times 2 = 4\pi i$.

4. Conclusion

The article systematically explains the use of residue theorem in the integration of trigonometric functions. Through variable substitution, convert real integrals into complex function contour integrals on unit circle, then use the residue theorem to solve. It is concerned with the correct recognition of isolated singularity in the unit circle and computing the residues. In both cases, the Laurent expansion of a power function is invoked when the pole is a simple singularity and of higher orders. In cases where trig appears in numerator, denominator, change them to be in the form of complex numbers, using trig identity based on Euler formula, and the poles could be readily determined with the corresponding orders. In the case of the complex trig function in the numerator and simple pole in the denominator of a complex contour, the removable singularity must not be thought of. When the numerator and denominator differ in the values of the behavior at certain poles, but agree at others on the contour, only consider the poles to obtain the residue, which is the coefficient of the poles negative one power term in the Laurent series of the complex trig function.

Two broad ways of calculating residue could be provided, using algebraic derivations, and are applicable to a large class of common cases, including normalized tangent function integral and rational trigonometric integral, as the following illustrate, in which such complex functions involving trigonometry were illustrated. In the case of parametric trig function, factor out the roots and convert the entire function to complex numbers. Just choose the root within the selected contour and compute its remainder. It is also a necessary step at first to distinguish between a pole and removable singularity. Removable singularities are not supposed to be counted in calculating residues. Then, with a complicated expression, where there are trigonometric expressions in both the numerator and the denominator, once it has been turned into a complex rational expression, be

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careful to make use of the order of each pole, and use different methods of calculating residues separately. Depending on the universality and efficiency of the residue theorem, research may further continue to determine how the residue theory may be applied to other forms of real integrals like exponential, logarithmic and improper definite integrals, and complemented by numerical methods to find the roots of higher-degree equations, to exploit more fully its capabilities in the area of complex analysis.

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