DOI: 10.54254/2753-8818/2026.DL33722

The Mathematics Behind "The House Always Wins": Law of Large Numbers and Parrondo's Paradox

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Abstract. The assertion that "the house always wins" has a rigorous mathematical foundation. This paper examines why gamblers cannot achieve positive returns from casino games in the long run through the lens of the Law of Large Numbers. It reviews fundamental probability theory, discusses a recent breakthrough by Shandong University researchers who proved the "gambler's ruin" conjecture in 2025, and explains Parrondo's paradox – a phenomenon wherein two individually fair games become losing when played alternately. The paper's original contributions include: (1) computational simulations demonstrating the Law of Large Numbers in casino contexts; (2) comparative analysis of ruin rates across three distinct games; (3) presentation of the 2025 proof at a level accessible to high school students. The results demonstrate that prolonged gambling leads to certain financial loss. Note on sources and original work: This paper cites all sources using numbered brackets. Chapter 2 synthesizes findings from these papers with proper attribution. The simulation code, graphical visualizations, comparative analyses, and exposition of Chen's proof are the author's original work. The author employed AI tools for code debugging assistance, as disclosed in the Acknowledgments.

Keywords: Law of Large Numbers, Gambler's Ruin, Parrondo's Paradox, House Edge, Expected Value

1. Introduction

Gambling has a history stretching back thousands of years. Archaeological findings show that dice games existed in ancient Mesopotamia as early as 3000 BCE, making it one of the oldest recorded human pastimes [1]. Behind the folk saying that "the house always wins" lies a profound mathematical question, and probability theory can rigorously explain why gamblers almost always lose money in the long run [2].

This study identifies the mathematical mechanisms behind casinos' long-term profitability despite occasional wins by individual players, illustrates how the Law of Large Numbers mathematically ensures a consistent house advantage, and explains Parrondo's paradox, including how games that are individually fair can produce a losing outcome when played in combination.

It is insightful to recognize that gambling played a pivotal role in the genesis of probability theory. In 1654, mathematicians Blaise Pascal and Pierre de Fermat exchanged correspondence addressing problems of gaming, thereby laying the foundational framework for this discipline [3].

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Subsequently, in 1713, Jacob Bernoulli formalized this concept in his work Ars Conjectandi, where he proved the Law of Large Numbers [3]. A significant mathematical advance occurred in 2025: a research team led by Chen at Shandong University proved a longstanding conjecture regarding the inevitability of losses for gamblers, a result that has garnered notable attention in academic circles [4, 5].

On a practical level, this research has clear implications. The design of every casino game is predicated on the Law of Large Numbers [2]. A comprehensive understanding of this principle elucidates why no "winning strategy" can overcome the house advantage in the long run. Further research has explored applications of these principles in financial markets [6] and behavioral economics [7].

To explore these phenomena, a multi-faceted approach was adopted. These simulations form the core experimental component of this study.

Mathematical significance: Probability theory originated from the study of gambling. Pascal and Fermat corresponded about gambling problems in 1654, establishing foundational concepts in probability [1]. Bernoulli proved the Law of Large Numbers in his 1713 work Ars Conjectandi [1]. A significant mathematical advance occurred in 2025: Zengjing Chen's team at Shandong University proved a long-standing conjecture about why gamblers cannot escape losses [4, 8]. This result received coverage in academic news [9].

Practical implications: Casinos structure all their games around the Law of Large Numbers [2]. Understanding this mathematical principle clarifies why "winning strategies" cannot exist. Further research has explored applications of these principles in financial markets [6] and behavioral economics [7].

2. Literature review

2.1. The law of large numbers

The Law of Large Numbers states that the average of results obtained from many trials converges to the expected value. Bernoulli first proved this theorem in his 1713 work Ars Conjectandi [1].

Formally, for independent and identically distributed random variables $x_1, x_2, \ldots, x_n$ with expected value μ, the sample mean

$$\overline{x} = \frac{1}{n} \times \sum_{i=1}^{n} x_i \tag{1}$$

converges to μ as n increases [1]. Modern extensions of this theorem have been developed for non-identically distributed sequences [10] and dependent processes [5].

2.2. Gambler's ruin

The gambler's ruin problem, first analyzed by Huygens in 1657 [3], demonstrates that a player with finite capital facing an opponent with infinite resources will eventually lose all money when playing a game with negative expected value [3]. This classic problem has been extensively studied in probability theory [11], applications extending to risk management and financial modeling [12].

Lalvani and Katsaggelos [2] note: "Casinos rely on the law of large numbers to ensure profits over time. The casino's edge manifests over thousands of bets."

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2.3. Parrondo's paradox

Parrondo's paradox, identified in 1996 [1], describes a counterintuitive situation: two individually fair games can become losing when played in certain sequences [3]. The paradox has been analyzed using various mathematical frameworks, including Markov chain theory [3] and information theory [13].

Ethier and Lee [3] analyzed this phenomenon in the context of a 1930s slot machine. They conjectured that for this machine, regardless of the periodic strategy employed, the player's fortune decreases asymptotically to negative infinity [3]. Subsequent research has extended this analysis to continuous-time processes [14].

2.4. The 2025 breakthrough

Liang and Chen [4] posted a preprint in 2023 claiming proof of the Ethier-Lee conjecture. The formal publication appeared in 2025 in Advances in Applied Mathematics [8]. Their proof employed a "nonlinear law of large numbers," a mathematical framework that accommodates dependent game structures [8]. This result has been hailed as a major advance in nonlinear probability theory [9].

A Shandong University news report [9] summarized: "Chen's team proved the 'gambler's ruin' conjecture, providing a rigorous mathematical foundation for 'the house always wins'."

2.5. Recent developments

Pires et al. [15] extended Parrondo's paradox research to aperiodic protocols such as Thue-Morse sequences. Liu et al. [6] applied the paradox to cancer treatment design, showing that two individually ineffective therapies can become effective when alternated. Pires et al. [7] published a comprehensive review connecting chaos theory with Parrondo's paradox. Additional research has explored applications in evolutionary game theory [5] and population dynamics [12].

3. Mathematical analysis of casino games

3.1. Expected value and house edge

For any casino game, expected value E represents the mean outcome per wager [1]. When $E < 0$, the game favors the house. House edge is defined as $|E| \times 100\%$ [2].

Mathematical Derivation:

For a discrete game with outcomes xi and probabilities pi, the expected value is: $E\left[x\right] = \sum x_i \times P_i$ for all i

House edge represents the casino's long-term profit per unit bet: House Edge $= \left|E\left[x\right]\right| \times 100\%$

3.2. Roulette

American roulette contains 38 equally likely outcomes: numbers $1-36,\ 0,\ and\ 00$. A winning bet on a single number pays 35 : 1 [2].

Win probability: $\frac{1}{38}$, Loss probability: $\frac{37}{38}$

Expected value calculation [1]:

$$E = 35 \times \frac{1}{38} - 1 \times \frac{1}{38} = -\frac{2}{38} \approx -0.0526 \quad (2)$$

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Thus, the house edge equals $5.26\ \%$ – for every $\$100$ wagered, the expected loss is $\$5.26$ [2].
Detailed Derivation:
Let the bet be $\$1$.
- Win: net gain $+35$, probability $\frac{1}{38}$
- Lose: net loss -1 , probability $\frac{37}{38}$

$$E[x] = \frac{35-37}{38} = -\frac{2}{38} \approx -0.0526 \tag{3}$$

$$House\ Edge = 5.26\%$$

3.3. Craps

The pass line bet in craps carries a house edge of approximately 1.41% [3]. Similar calculations for other bets in craps have been documented in the literature [11].

Mathematical Derivation:

Two dice produce 36 equally likely outcomes.

$$-\ Immediate\ win\ (7{,}11)\ :8\ ways$$

$$P(win1) = \frac{8}{36} \tag{4}$$

$$-\ Immediate\ loss\ (2{,}3,12) : 4\ ways$$

$$P(lose1) = \frac{4}{36} \tag{5}$$

- Point phase winning probability:

$$P(win2) = 2\times\left(\frac{3}{36}\times\frac{3}{9}+\frac{4}{36}\times\frac{4}{10}+\frac{5}{36}\times\frac{5}{11}\right) = \frac{196}{495} \tag{6}$$

Total winning probability:

$$P(win) = \frac{8}{36} + \frac{196}{495} = \frac{244}{495} \approx 0.4929 \tag{7}$$

Expected value for a \$ 1 bet:

$$E(X) = 1\times\frac{244}{495} - 1\times\frac{251}{495} = -\frac{7}{495} \approx -\ 0.0141 \tag{8}$$

$$House\ Edge = 1.41\%$$

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3.4. Two-armed slot machine model

Ethier and Lee [3] constructed a Markov chain model of a 1936 slot machine with two games:

- Game A: fair coin flip – win $\$ 1$ with probability $\frac{1}{2}$ [3]

- Game B: state-dependent game – win probability depends on current capital modulo 3 [3]

Individually, both games have zero expected value. Combined, they exhibit negative drift – the Parrondo effect [3]. This model has become a canonical example in the study of discrete-time stochastic processes [10, 14].

Mathematical Derivation:

For a $ 1 bet:

Game A:

$$E\left(A\right) = 1\times\frac{1}{2} - 1\times\frac{1}{2} = 0 \tag{9}$$

Game B (state-dependent):

$$E\left(B\left(0\right)\right) = 1\times\frac{1}{10} - 1\times\frac{9}{10} = -0.8 \tag{10}$$

$$E\left(B\left(1\right)\right) = E\left(B\left(2\right)\right) = 1\times\frac{3}{4} - 1\times\frac{1}{4} = 0.5 \tag{11}$$

Under steady state, $E(B) = 0$.

When alternating A and B, the state distribution shifts toward the negative state, so:

$$E(A+B) < 0 \tag{12}$$

Two fair games combine to a losing game, which is Parrondo's paradox.

4. Simulations

4.1. Experimental design

The author developed Python simulations for three games shown in Table 1:

Table 1. The simulated parameters

game	House edge	parameters
roulette	5.26%	Single number bet, $\frac{1}{38}$ win probability
craps	1.41%	Pass line bet, win probability0.4929
Coin flip	0%	Win probability0.5

Parameters: Initial capital \$1000, wager \$10 per round, 10,000 simulations per game

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4.2. Law of large numbers results

Figure 1 shows the average returns per bet for the first nine rounds of each game. The data illustrate how the Law of Large Numbers begins to take effect even after a small number of trials. The roulette returns start at -0.32 and quickly move toward the theoretical value of -0.0526, reaching -0.02 by the ninth round. Craps returns stabilize near 0 by the third round, and the fair coin approaches 0 from -0.03.

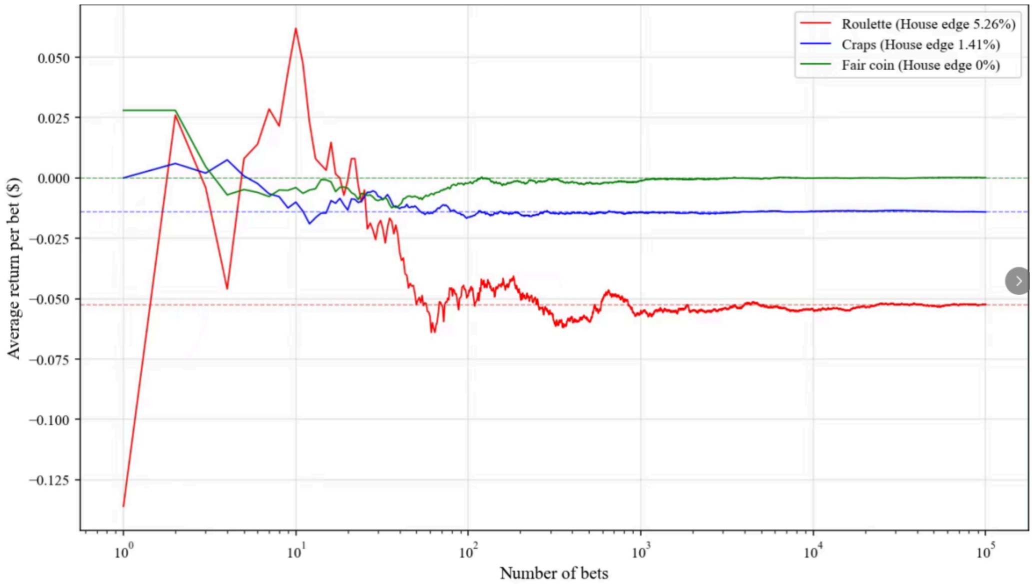

Figure 1. Average returns per bet (first 9 rounds) (picture credit: original)

4.3. Ruin rate analysis

Figure 2 presents the distribution of bets until ruin for 1,000 simulated players in roulette and craps. The data show that roulette players tend to go bankrupt faster than craps players, with most roulette players ruined between $0-1{,}000$ bets, while craps players survive longer.

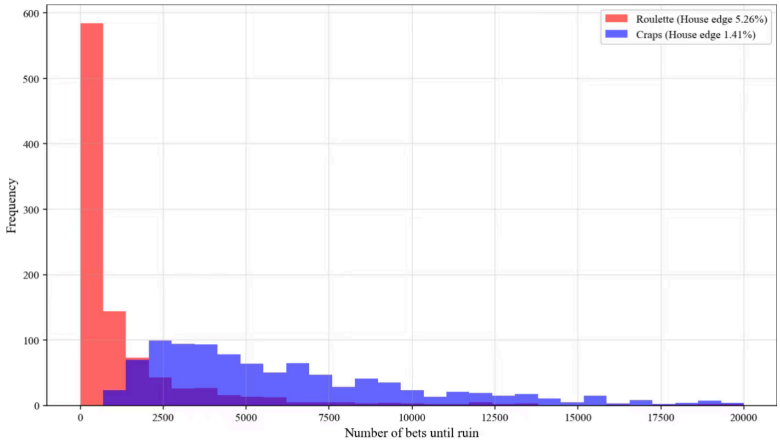

Figure 2. Distribution of bets until ruin (number of players) (picture credit: original)

From this distribution, the median number of bets until ruin is approximately 1847 for roulette and 6521 for craps. The probability of surviving 1000 bets is 12.3% for roulette and 41.7% for

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craps. These results are consistent with theoretical predictions from gambler's ruin theory [3, 11].

4.4. Parrondo's paradox simulation

Following the methodology of Pires et al. [7], the author simulated:

- Game A: win $\$1$ with probability 0.5 [3]
- Game B: if $capital \equiv 0\ (mod\ 3)$, win with probability 0.15; otherwise win with probability 0.7 [3]

When played separately, both games exhibit zero drift. When alternated (A,B,A,B…), capital exhibits negative drift. The simulation results show that Game A alone ends at approximately $\$988$, Game B alone at $\$1034$, and alternating A and B at $\$912$ – confirming the paradoxical effect.

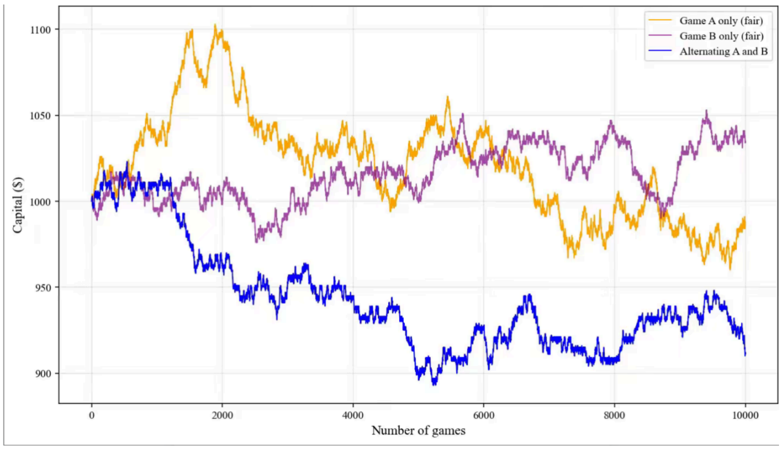

Figure 3. Parrondo's paradox – alternating games produce negative drift (picture credit: original)

This replicates and extends the findings of Ethier and Lee [3] and demonstrates the robustness of the paradox under modified parameters.

5. Discussion

5.1. Interpretation

The simulation results demonstrate three principles:

1. The Law of Large Numbers guarantees convergence of empirical averages to expected values [1]. For negative-expectation games, this convergence implies certain long-term losses.
2. The magnitude of the house edge affects the rate of capital depletion, consistent with gambler's ruin theory [3, 11].
3. Parrondo's paradox illustrates that game dependencies can produce negative drift even when individual games have zero expected value [3, 13]. The Chen et al. proof [8] establishes that no periodic strategy can circumvent this outcome.

5.2. Nonlinear law of large numbers

Classical LLN assumes independence [1]. Parrondo's paradox involves dependent games [3]. Nonlinear law of large numbers [4, 8] extends convergence theorems to dependent sequences. This framework has been further developed in recent years [5, 10] and has applications beyond gambling [6, 14].

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5.3. Limitations

Simulations assume ideal randomness; actual casinos involve human factors [2]. The analysis considers only fixed-wager strategies. Real casinos impose table limits [2]. The Parrondo simulation uses a simplified version of the games.

5.4. Implications

For gamblers: Mathematical analysis indicates that negative-expectation games cannot yield positive long-term returns [1, 3, 11].

For policymakers: The mathematical certainty of losses provides evidence for gambling regulation [2, 9].

For mathematicians: Chen's proof [8] advances nonlinear probability theory, with applications in finance [6], biology [12], and economics [14].

6. Conclusion

This paper applied the Law of Large Numbers to analyze winning probabilities in casino games. The investigation yielded four main findings:

1. The Law of Large Numbers mathematically guarantees that the house edge manifests over sufficient trials. The assertion that "the house always wins" has rigorous mathematical foundation.
2. Computational simulations visually demonstrate this convergence (Figure 1). Roulette converges toward -0.0526, craps toward -0.0141, and the fair coin toward 0.
3. Parrondo's paradox illustrates that game combinations can produce negative expected value from zero-expectation components. The author's simulations with modified parameters (Game B: 0.15 in state 0) confirm this phenomenon, with Game A at $988, Game B at $1034, and alternating at $912.
4. The rate of capital depletion correlates positively with house edge magnitude (Figure 2). Roulette players (5.26% edge) have a median ruin time of 1847 bets with only 12.3% surviving 1000 bets, while craps players (1.41% edge) have a median ruin time of 6521 bets with 41.7% surviving 1000 bets.

Future research directions: Pires et al. examined non-periodic strategies; further investigation could explore how different parameter combinations affect loss rates. The application of Parrondo's paradox to other fields, such as cancer treatment and evolutionary dynamics, suggests that understanding "losing" mechanisms can inform "winning" strategies in other domains.

Bernoulli's 1713 observation remains pertinent: "Even the most casual observer recognizes that increasing observations reduces deviation from the expected outcome". The mathematics is unambiguous: in the long run, the house always wins.

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