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Research on Cauchy Integral Theorem, Formula and Their Applications

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Abstract. In complex analysis, Cauchy integral theory has been placed at the heart. In particular, the Cauchy Integral Formula is recognized as a powerful extension of the namesake theorem that overcomes the problem of isolated singular points in complex contour integrals, and indicates the fundamental connection between the values on the boundary and internal values of analytic functions. This paper follows the Cauchy Integral Theorem as it systematically expounds on the statements, rigorous derivation procedures and applicable conditions of the Cauchy Integral Formula and its higher-order derivative formula which is an extension. Together with common complicated examples, it studies the practical uses in the calculation of high-order singular integrals and real improper integrals. Meanwhile, it summarizes the theoretical applications in proving properties of analytic functions, and practical uses in physics and engineering. The work classifies types of out logical systems of complex integral calculation, demystifies significant uses of the theory of the Cauchy integral, and offers credible sources of related mathematical research and applications issues across disciplines.

Keywords: Cauchy integral formula, analytic function, contour integral, higher-order derivative formula

1. Introduction

Complex analysis, an important field of modern mathematics, focuses on the study of analytic functions and integration of complex contours, establishing a comprehensive theoretical framework that theorizes and unites pure mathematical investigations with practical use in engineering. The Cauchy Integral Theorem, as the mainstay of complex analysis, provides the basic properties of contour integrals of analytic functions of simply connected domains, and forms a sound basis of simplifying the computation of traditional complex integrals without singular points. It does however have apparent shortcomings: it does not explicitly address complex integrals with integrands that have isolated singularities, which are very common in mathematical analysis, physical modelling problems, engineering calculation problems, and therefore limits the further formulation and use of the complex integral theory [1, 2].

In order to overcome this limitation, Augustin-Louis Cauchy proposed to the original integral theorem the Cauchy Integral Formula. This formula does not only accomplish the unity of solution of complex integrals with single isolated singularities, but it also demonstrates the fundamental property of analytic functions: the inner values of a domain are entirely defined by the outer ones. In addition, the higher-order derivative formula, based on the Cauchy Integral Formula, is another solution to the higher-order singularity of integrals, and is an infinitely differentiable analytic function, which is a property unshared by real functions [3, 4].

In the above theoretical background, this paper methodically examines the statements of the theorems, rigorous derivation, conditions and fundamental corollaries of the Cauchy integral theorem, formula and its higher-order derivative counterpart. Together with numerous detailed typical examples, it delves into the particular application techniques of these theories to calculating complex contour integrals and real improper integrals and clears up their value in application to the proof of analytic functions properties and engineering applications. The purpose of this paper is to elucidate the fundamental reasoning behind the Cauchy integral system and give a clear theoretical and practical point of reference to the study and practice of complex integral calculation.

2. Theorem and formula

2.1. Statement of the cauchy integral theorem

2.1.1. Simply connected domain scenario

Theorem (Cauchy–Goursat Theorem). Assumed that D is a simply connected region in the complex space, and $f\,(z)$ be analytic on every point in D [5]. Therefore, for any simple closed loop C which fully inside of D, including both interior and border portions also lying inside of D convolution integrated over this contour can find the following solution:

$$f\,(\mathrm{z}_0) = \frac{1}{2\pi} \oint_{\mathrm{C}} \frac{\mathrm{f(z)}}{\mathrm{z}-\mathrm{z}_0}\,\mathrm{dz} \tag{1}$$

The Domain Should Be simply connected. This is what makes the contour C be continuously contractible to a single point within the domain, without any intervals or singular points. When the domain is such that the interval is not empty, the integral of the contour around such a hole can be a non-zero value, in spite of the fact that the same contour may be in the non-hole domain of an analytic function. The formula needs to be analytic in the entire domain. The essence of the theorem is analyticity. A simple analytic function is differentiable at every point of the domain, and hence, satisfies the Cauchy-Riemann equations. Then to cancelation of integral values round curves. In the event that the function contains singular points, which are non-analytic points of the domain, then the direct application of the theorem is not possible. The Contour Should Be a Simple Closed Curve in the Domain. The fact that it is simple implies that it has no points of self-intersection, and the fact that it is closed means that the initial point coincides with the final point. In the case where the contour crosses or goes outside the analytic domain, the result of the integral is not zero as per the theorem. The variation of a smooth function is conserved in a simply connected domain. Any closed path traversed in the domain causes all the increments of the integrals to be canceled out and the final integral to be equal to zero.

2.1.2. Multiply connected domain scenario theorem

Supposed that D is a multiply connected regions with an outer simple closed loop C, and n is mutually disjoint, non-inclusive inner contours $\mathrm{C}_1, \mathrm{C}_2 \ldots\ldots C_n$ located inside C [6].

If the function f(z) is analytic at every point in the interval between C and each inner contour $\mathrm{C_k}$, the integral relationship is given by:

$$\oint_{\mathrm{C}} \mathrm{f}\,(\mathrm{z})\mathrm{dz} = \oint_{C_1} \mathrm{f}\,(\mathrm{z})\mathrm{dz} + \oint_{C_2} \mathrm{f}\,(\mathrm{z})\mathrm{dz} + \cdots + \oint_{C_2} \mathrm{f}\,(\mathrm{z})\mathrm{dz} \tag{2}$$

All contours take the counterclockwise direction as their positive orientation.

If all inner contours are traversed in the clockwise direction, they form a composite contour, and the integral along this combined contour satisfies:

$$\oint_{\mathrm{C}-\mathrm{C}_1-\mathrm{C}_2-\ldots-\mathrm{C}_\mathrm{n}} \mathrm{f}\,(\mathrm{z})dz = 0 \tag{3}$$

It is possible to transfer a multiply connected region to a simply one by making crosscuts between the exterior and interior boundaries. Integrals of crosscut directions cancel each other along these crosscuts, and only the relation of integrals between the outer and inner boundaries persists. In this way, this generalizes the Cauchy Integral Theorem from simply connected regions to multiply ones.

2.2. Fundamental corollaries

If $f(z)$ is analytic in a simply connected domain D, then the integral above does not depend on the path from z_0 to z_1 . When $f(z)$ is analytic on a simply connected region D , the value of the integral above remains the same regardless of which path connects z_0 and z_1 . The formulate $\int_{\mathrm{z}_0}^{\mathrm{z}_1} \mathrm{f}\,(\mathrm{z})\mathrm{dz}$ is determined solely by the initial and final points, no matter what the shape of the path connecting the two points [7].

Take two arbitrary paths C_1 and C_2 connecting z_0 and z_1 ; the combination of C_1 and the reversed path $-\mathrm{C}_2$ forms a closed path. In light of the Cauchy Integral Theorem, it is arrived that $\int_{\mathrm{C}_1} \mathrm{f}\,(\mathrm{z})\mathrm{dz} - \int_{\mathrm{C}_2} \mathrm{f}\,(\mathrm{z})\mathrm{dz} = 0$.

If $f(z)$ is analytic on a simply connected domain D , then $f(z)$ possesses a primitive (antiderivative) defined on D — specifically, an analytic function $F(z)$ satisfying $F'(z) = f(z)$. Such a primitive can be expressed by means of an integral with a variable upper limit.

$$F\,(z) = \int_{\mathrm{z}_0}^{\mathrm{z}_1} \mathrm{f}\,(\mathrm{z})\mathrm{dz} \tag{4}$$

Because the upper limit integral is irrelevant of the path taken, this definition is proper and is independent of the choice of path. More evidence that F(z) is analytic in D The equality of F(z) and its derivative is still more evidence of the fact that F(z) is analytic in D. This is the existence part of the basic theorem of calculus in complex analysis.

Let $f(z)$ be analytic over a simply connected region D , and denote by $F(z)$ a primitive function of $f(z)$. For any point, the integral formula holds:

$$\int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_2) \tag{5}$$

It is similar to real function calculus, definite integrals of complex analytic functions can be computed directly using primitive function values at endpoints, this eliminating the need for path parameterization. This greatly simplifies the calculation of integrals along non-closed contours [8].

If $f(z)$ can be analytic in the regimes sandwiched between two simple closed contours C_1 and C_2 , the integral equality is as follows:

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz \tag{6}$$

The integral value remains unchanged during continuous contour deformation, provided no singular points are crossed in the process. This implies three key conclusions. Integrals over large circles can be substituted with integrals over small circles. Circular contours can be replaced by squares, triangles, or other arbitrary shapes. The integral just depends on enclosed singular points, not the contour's shape.

2.3. Cauchy integral formula

Assume D can be simply connected on the complex plane, D has a contour, let C denote any positively oriented simple closed contour, the function f(z) is analytic in D, and z_0 is any point in the contour C. Thus, the Cauchy integral formula should have the behavior of:

$$f(z_0) = \frac{1}{2\pi} \oint_C \frac{f(z)}{z-z_0} dz \tag{7}$$

An equivalent form is $2\pi f(z_0) = \oint_C \frac{f(z)}{z-z_0} dz$.

Proof: Using the contour deformation principle, shrink the large contour into a tiny circle in the neighbor of the singular point z_0 and derive the formula through limit operation.

Construct a small circle: Take z_0 as the center and draw a positively oriented small circle $C_\epsilon |z-z_0| = \epsilon$ such that C_ϵ is completely contained inside C. Since $\frac{f(z)}{z-z_0}$ is smooth in the regime between C and C_ϵ , by the contour deformation principle $\oint_C \frac{f(z)}{z-z_0} dz = \oint_{C_\epsilon} \frac{f(z)}{z-z_0} dz$. Then, Split the integrand: Decompose f(z) into $f(z_0) + [f(z) - f(z_0)]$, then the integral can be split into two parts: $\oint_{C_\epsilon} \frac{f(z)}{z-z_0} dz = f(z_0) \oint_{C_\epsilon} \frac{1}{z-z_0} dz + \oint_{C_\epsilon} \frac{f(z)-f(z_0)}{z-z_0} dz$. After that, Calculate the first integral: From the basic conclusion of complex integrals, $\oint_{C_\epsilon} \frac{f(z)}{z-z_0} dz = 2\pi$, so, the value of the first part is $2\pi i f(z_0)$[9].

Last, analyze the second integral: Since f(z) can be smooth at z_0 ,and f(z) is continuous at z_0 . For any $\varepsilon > 0$, exist $\delta > 0$, as $|z - z_0| = \epsilon < \delta$, there is $|f(z) - f(z_0)| < \varepsilon$. by the integral estimation inequality:

$$\left|\oint_{C_\epsilon}\frac{f(z)-f(z_0)}{z-z_0}dz\right| \le \frac{\varepsilon}{\epsilon}\cdot 2\pi\epsilon = 2\pi\varepsilon \tag{8}$$

In summary, take the limit, one obtains that

$$\oint_C \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0) \tag{9}$$

and rearranging it gives the Cauchy integral formula.

The Cauchy integral formula can be directly generalized to arbitrary-order derivatives of analytic functions, proving that analytic functions are infinitely differentiable and all their higher-order derivatives are still analytic, which is a core property not possessed by real functions. The statement of the higher-order derivative formula is:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz \tag{10}$$

where $n = 1,2,3,\ldots$ and $f^{(n)}(z_0)$ denotes the n -th order derivative of $f(z)$ at z_0 . This formula is a core tool for calculating complex integrals containing higher-order singularities and a direct basis for proving the infinite differentiability of analytic functions.

3. Practical applications of the cauchy integral formula

Example 1. Compute integral $\oint_C \frac{e^{z^2}}{z-2}dz$ where the curve of C is centered at $z = 1$.

Solution: Given the entire function $f(z) = e^{z^2}$, and letting C be a positively oriented simple closed curve that encloses the point $z = 2$, Cauchy's integral formula implies that the integral is $2\pi i\, f(2)$, which simplifies to $2\pi i e^4$.

Example 2. Compute the integral in the same manner as the earlier examples, now with C representing the curve shown.

Solution: This one is trickier. Let f(z) = e^{z^2} . The path taken around two includes both curves on either side of function continuation C going around once in the clockwise direction [10]. Since both are simple closed curves, the Cauchy integral formula can be applied separately to each one. The integrals yield negative values because the curves are traversed clockwise around $z = 2$.

$$\oint_C \frac{f(\varphi)}{\varphi-2}d\varphi = \oint_{C_1}\frac{f(\varphi)}{\varphi-2}d\varphi + \oint_{C_2}\frac{f(\varphi)}{\varphi-2}d\varphi = -2\pi i f(2) - 2\pi i f(2) = -4\pi i f(2) \tag{11}$$

Example 3. Calculate the integral $\oint_C \frac{\cos z}{(z-\pi/2)^3}dz$ in which C is the positively oriented circle of the form $\left|z-\frac{\pi}{2}\right| = 1$.

The first step is to identify parameters. Given that the contour $C|z-\pi/2| = 1$ is a positively oriented simple closed curve, the singular point $z_0 = \frac{\pi}{2}$ lies inside C. From the integrand $\frac{\cos z}{(z-\pi/2)^3}$, this identify $f(z) = \cos z$, $n+1 = 3 \Rightarrow n = 2$. The next step, Compute derivatives: $f(z) = \cos z, f'(z) = -\sin z, f''(z) = -\cos z$ .The third step is Evaluating at $z_0 = \frac{\pi}{2}$. $f''\left(\frac{\pi}{2}\right) = -\cos\frac{\pi}{2} = 0$. The last step is Substitute into the formula $\oint_C \frac{\cos z}{(z-\pi/2)^3} dz = \frac{2\pi i}{2!} f''\left(\frac{\pi}{2}\right) = \frac{2\pi i}{2} \cdot 0 = 0$.

Example 4. Compute the integral $I = \oint_{|z|=5} \frac{z^3+2z+1}{(z-1)(z-3)^2} dz$.

Solution: There are two singular points inside $|z| = 5 : z = 1$ (first-order) and z=3 (second-order).Decompose the rational function into partial fractions: $\frac{z^3+2z+1}{(z-1)(z-3)^2} = \frac{1}{z-1} + \frac{5}{z-3} + \frac{28}{(z-3)^2}$ Divide the integral into three parts: $I = \oint_{|z|=5} \frac{1}{z-1} dz + \oint_{|z|=5} \frac{5}{z-3} dz + \oint_{|z|=5} \frac{28}{(z-3)^2} dz$. For z=1 (first-order singularity), In light of Cauchy's integral formula, $\oint_{|z|=5} \frac{1}{z-1} dz = 2\pi i\ 2$. For z=3, the first-order term: $\oint_{|z|=5} \frac{5}{z-3} dz = 5 \cdot 2\pi i = 10\pi i$. For z=3 (second-order singularity), by the higher-order derivative formula: $\oint_{|z|=5} \frac{28}{(z-3)^2} dz = 28 \cdot 2\pi i = 56\pi i$. Sum up all results: $I = 2\pi i + 10\pi i + 56\pi i = 68\pi i$.

4. Conclusion

The present work investigates Cauchy's integral theorem, Cauchy's integral formula, and the generalization of the latter to derivatives of arbitrary order. It makes clear the circumstances under which they are applied, the methods of their derivation, and their actual implications. Through typical examples, it shows that they work well for complex integrals, theoretical proofs, and engineering problems. This indicates that Cauchy's integral formula effectively fills the gap left by Cauchy's integral theorem and serves as a crucial bridge between complex analysis theory and real - world application, providing a key tool for complex integration and the study of analytic functions as one looks to the future. there's plenty of room to grow, like extending these ideas to multiply connected domains, multiple singularities, and fancier integral setups to make solving strategies better and enrich complex analysis theory, while on the applied side, and can team up with fields like signal processing, quantum mechanics, fluid mechanics, and aerospace engineering to bring Cauchy's integral formula into more real engineering problems and get better at handling complex integrals and boundary value issues — plus. When Cauchy's theory is combined with modern computational math and AI, it may generate some new ideas and impetus for quick complex integration, cross - disciplinary breakthroughs, and further advancement of mathematical knowledge.

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