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Geometric Generalization of Spectral Symmetry: From Normal Operators to Diagonalizable Operators

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Abstract. The adjoint relation of linear operators is a core structure in inner product spaces. Normal operators, due to their commutativity with their adjoints, possess elegant spectral properties. Classical theory shows that a normal operator and its adjoint share the same eigenvectors with conjugate eigenvalues. However, this conclusion is confined to the normal case and typically relies on the spectral theorem or triangularization techniques. This paper generalizes the idea of spectral symmetry to arbitrary diagonalizable operators and reveals its essence from a purely geometric perspective. For a diagonalizable operator on a finite-dimensional complex inner product space, we construct a new operator based on its eigenspace decomposition, which acts as the conjugate of the eigenvalues on each eigenspace. Furthermore, by constructing a new inner product that depends on the eigenspace decomposition, the original operator and this new operator become mutual adjoints under this new inner product, and their relationship with the standard adjoint is established via a similarity transformation. The positive self-adjoint operator introduced herein captures the geometric inclination of the eigenspaces with respect to the original inner product. When the operator is normal, this construction reduces to the classical case. This paper provides a unified geometric explanation of the connection between the spectral structure of diagonalizable operators and the adjoint relation, offering a new perspective for operator theory and matrix analysis.

Keywords: Normal operator, spectral symmetry, diagonalizable operator, generalized adjoint, inner product

1. Introduction

Spectral theory of linear operators is a central branch of functional analysis and linear algebra, where the relationship between an operator and its adjoint plays a fundamental role [1]. In finite-dimensional complex inner product spaces, normal operators (satisfying $AA^{*}=A^{*}A$) are well understood due to their unitary diagonalizability. The spectral theorem guarantees the existence of an orthonormal eigenbasis, with the adjoint operator having conjugate eigenvalues relative to this basis [1]. This structure underlies important applications in quantum mechanics and linear algebra [2,3]. However, existing proofs of spectral symmetry for normal operators often rely on the spectral theorem itself or on matrix decomposition techniques, offering limited geometric insight. More importantly, for general diagonalizable but non-normal operator, eigenspaces are no longer orthogonal, and the adjoint operator preserve a simple conjugate spectral correspondence. This raises a natural question: can one define a generalized adjoint that restores such symmetry, and what is its geometric relation to the standard adjoint? [4].

This paper addresses these questions by extending spectral symmetry to arbitrary diagonalizable operators through a geometric framework. This work provides a geometric interpretation of the spectral symmetry of normal operators and naturally extends it to general diagonalizable operators, forming a unified theory of spectral symmetry. It reveals the deep interplay between the spectral structure of diagonalizable operators and the geometry of the inner product, offering new geometric tools for further studies on perturbation analysis of non-normal operators, numerical stability, and non-Hermitian quantum mechanics. Moreover, the construction of the new inner product itself has independent mathematical value and can be applied to other classification and geometric problems concerning operators.

2. Spectral symmetry of normal operators and its generalization

2.1. Spectral symmetry of normal operators

Let A be a linear operator defined on a finite-dimensional unitary space V over a field F, with dimension n. Denote the distinct eigenvalues of A by $\lambda_1,\lambda_2,\ldots,\lambda_t$, and let the algebraic multiplicity of λ_i be n_i ($i=1,2,\ldots,t$). Then the adjoint operator A^* satisfies [5]:

$$\varphi_{A^*}(\lambda)=\det\left(\lambda I-A^*\right)=\det\left(\lambda\overline{I}-\overline{A}\right)=\overline{\varphi}_A(\lambda) \tag{1}$$

where φ_A and φ_{A^*} are the characteristic polynomials of A and A^*. This shows that the distinct eigenvalues of A^* are $\overline{\lambda_i}$ with the same algebraic multiplicities n_i; hence $\sum_{i=1}^{t} n_i=n$.

When A is normal, we have $AA^*=A^*A$[6]. Then each eigenspace V_{λ_i} of A is an invariant subspace of A^*[7]. Take any eigenvector $\beta\in V_{\lambda_i}$ of the restriction $A^*|_{V_{\lambda_i}}$ with eigenvalue μ, i.e. $A^*|_{V_{\lambda_i}}\beta=\mu\beta$. By the property of the adjoint,

$$\mu(\beta,\beta)=(\beta,\mu\beta)=\left(\beta,A^*\beta\right)=(A\beta,\beta)=\overline{\lambda_i}(\beta,\beta)\Rightarrow\mu=\overline{\lambda_i} \tag{2}$$

Thus the only eigenvalue of $A^*|_{V_{\lambda_i}}$ is $\overline{\lambda_i}$. On the other hand, take an eigenvector $\alpha\in V_{\lambda_i}$ of A and orthogonally decompose $A^*\alpha$ as $A^*\alpha=u+v$ with $u\in F\alpha$ and $v\in(F\alpha)^{\perp}$. Then

$$\left(A^*\alpha,\alpha\right)=(\alpha,A\alpha)=(\alpha,\lambda_i\alpha)=\lambda_i(\alpha,\alpha)=\left(\overline{\lambda_i}\alpha,\alpha\right)\Longrightarrow\left(A^*\alpha-\overline{\lambda_i}\alpha,\alpha\right)=0 \tag{3}$$

which shows $A^*\alpha-\overline{\lambda_i}\alpha\in(F\alpha)^{\perp}$; by the uniqueness of the orthogonal decomposition we must have $\overline{\lambda_i}\alpha$. Using the property [8]:

$$\|A^*\alpha\|=\|A\alpha\| \tag{4}$$

since $\|A^*\alpha\|^2=(A^*\alpha,A^*\alpha)=(\alpha,AA^*\alpha)=(\alpha,A^*A\alpha)=(A\alpha,A\alpha)=\|A\alpha\|^2$), we obtain:

$$\Longrightarrow|\lambda_i|^2|\alpha|^2=|A\alpha|^2=\left|A^*\alpha\right|^2=|u|^2+|v|^2=|\lambda_i|^2|\alpha|^2+|v|^2\Longrightarrow v=0 \tag{5}$$

Hence α is also an eigenvector of A^* with eigenvalue $\overline{\lambda_i}$. By the spectral theorem, a normal operator A is diagonalizable, so $\dim V_{\lambda_i}=n_i$. Consequently, the characteristic polynomial of $A^*|_{V_{\lambda_i}}$ is $\varphi_{A^*|_{V_{\lambda_i}}}=(\lambda-\overline{\lambda_i})^{n_i}$. Since A^* is normal and thus diagonalizable, the restriction $A^*|_{V_{\lambda_i}}$ is also diagonalizable (its minimal polynomial divides that of A^* and has no repeated roots), so the geometric multiplicity of $\overline{\lambda_i}$ for $A^*|_{V_{\lambda_i}}$ is n_i . Therefore $n_i\leq\dim U_{\overline{\lambda_i}}$, where $U_{\overline{\lambda_i}}$ denotes the eigenspace of A^* for eigenvalue $\overline{\lambda_i}$ ($i=1,2,...,t$). But

$$V=U_{\overline{\lambda_1}}\oplus U_{\overline{\lambda_2}}\oplus...\oplus U_{\overline{\lambda_t}}\Rightarrow n=\sum_{i=1}^{t}n_i\leq\sum_{i=1}^{t}\dim U_{\overline{\lambda_i}}=\dim V=n \tag{6}$$

Hence we must have $n_i=\dim U_{\overline{\lambda_i}}$ for all i , and consequently $U_{\overline{\lambda_i}}=V_{\lambda_i}$. That is, each eigenspace of A corresponding to λ coincides with the eigenspace of A^* corresponding to $\bar{\lambda}$; every eigenvector of A (for eigenvalue λ) is also an eigenvector of A^* (for eigenvalue $\bar{\lambda}$). Thus in a unitary space, a normal operator and its adjoint share all eigenvectors and exhibit conjugate spectral symmetry.

2.2. Generalized adjoint operator

In fact, one can define a conjugate-spectral-symmetric operator for every operator. Let A be a diagonalizable linear operator on a finite-dimensional unitary space V over F . Define an operator B such that for every eigenvector β of A with $A\beta=\lambda\beta$, we have $B\beta=\bar{\lambda}\beta$. Since a diagonalizable operator A possesses a basis of eigenvectors, B is uniquely determined by this definition and enjoys the following properties.

B commutes with A : $AB=BA$. Take any $v\in V$. Because A is diagonalizable, the eigenspaces V_{λ_i} ($i=1,2,...,t$) satisfy $V=\bigoplus_{i=1}^{t}V_{\lambda_i}$. Write the unique decomposition $v=\sum_{i=1}^{t}v_i$ with $v_i\in V_{\lambda_i}$. Then

$$A\,(Bv)=A\sum_{i=1}^{t}\overline{\lambda}_iv_i=\sum_{i=1}^{t}\overline{\lambda}_iAv_i=\sum_{i=1}^{t}\overline{\lambda}_i\lambda_iv_i \tag{7}$$

$$B\,(Av)=B\sum_{i=1}^{t}\lambda_i v_i=\sum_{i=1}^{t}\lambda_i Bv_i=\sum_{i=1}^{t}\lambda_i\overline{\lambda}_i v_i \tag{8}$$

Thus $A(Bv)=B(Av)$ for all v, so $AB=BA$.

(2) B commutes with every operator C that commutes with A. Since $AC=CA$, each eigenspace V_{λ_i} is invariant under C. The restriction $C|_{V_{\lambda_i}}$ obviously commutes with the scalar operators $A|_{V_{\lambda_i}}$ and $B|_{V_{\lambda_i}}$. Because V is the direct sum of the V_{λ_i}, it follows that C commutes with B on the whole space.

2.3. Definition of a new inner product

Based on the above decomposition, we define a new inner product on V. For any $x,y\in V$ with unique decompositions $x=\sum_{i=1}^{t}x_i$, $y=\sum_{i=1}^{t}y_i$ ($x_i,y_i\in V_{\lambda_i}$), set

$$x=\sum_{i=1}^{t}x_i,\qquad y=\sum_{i=1}^{t}y_i,\qquad x_i,y_i\in V_{\lambda_i} \tag{9}$$

where $\langle x, y\rangle=\sum_{i=1}^{t}(x_i, y_i)$ denotes the original inner product. It is straightforward to verify that this defines an inner product:

Conjugate symmetry:

$$\langle x, y\rangle=\sum_{i=1}^{t}(x_i, y_i)=\sum_{i=1}^{t}\overline{(y_i, x_i)}=\overline{\sum_{i=1}^{t}(y_i, x_i)}=\overline{\langle y, x\rangle} \tag{10}$$

Conjugate bilinearity:

$$\langle x+z, y\rangle=\sum_{i=1}^{t}(x_i+z_i, y_i)=\sum_{i=1}^{t}(x_i, y_i)+\sum_{i=1}^{t}(z_i, y_i)=\langle x, y\rangle+\langle z, y\rangle$$

$$\langle kx, y\rangle=\sum_{i=1}^{t}(kx_i, y_i)=\overline{k}\sum_{i=1}^{t}(x_i, y_i)=\overline{k}\,\langle x, y\rangle$$

$$\langle x, y+z\rangle=\sum_{i=1}^{t}(x_i, y_i+z_i)=\sum_{i=1}^{t}(x_i, y_i)+\sum_{i=1}^{t}(x_i, z_i)=\langle x, y\rangle+\langle x, z\rangle$$

$$\langle x, ky\rangle=\sum_{i=1}^{t}(x_i, ky_i)=k\sum_{i=1}^{t}(x_i, y_i)=k\,\langle x, y\rangle \tag{11}$$

Positive definiteness:

$$\langle x, x\rangle=\sum_{i=1}^{t}(x_i, x_i)=\sum_{i=1}^{t}|x_i|^2\geq 0,\qquad \langle x, x\rangle=0\Longleftrightarrow |x_i|=0,\ i=1,2,\ldots t\Longleftrightarrow x=0 \tag{12}$$

Under this new inner product, A and B are mutual adjoints:

$$\langle Ax,y\rangle=\left\langle A\sum_{i=1}^{t}x_i,y\right\rangle=\left\langle\sum_{i=1}^{t}Ax_i,y\right\rangle=\left\langle\sum_{i=1}^{t}\lambda_i x_i,y\right\rangle=\overline{\lambda}_i\sum_{i=1}^{t}\langle x_i, y\rangle=\overline{\lambda}_i\sum_{i=1}^{t}(x_i, y_i) \tag{13}$$

$$\langle x, By\rangle=\left\langle x, B\sum_{i=1}^{t}y_i\right\rangle=\left\langle x, \sum_{i=1}^{t}By_i\right\rangle=\left\langle x, \sum_{i=1}^{t}\overline{\lambda}_i y_i\right\rangle=\overline{\lambda}_i\sum_{i=1}^{t}\langle x, y_i\rangle=\overline{\lambda}_i\sum_{i=1}^{t}(x_i, y_i) \tag{14}$$

Together with the commutativity of A and B, we obtain that A and B are normal operators with respect to the new inner product.

2.4. Relation between the generalized adjoint and the standard adjoint

Using the direct sum decomposition $V=\bigoplus_{i=1}^{t}V_{\lambda_i}$ and the unique representation $v=\sum_{i=1}^{t}v_i$ ($v_i\in V_{\lambda_i}$), define linear operators $P_i:V\to V$ as the projections onto V_{λ_i} ; i.e., for $v=\sum_{i=1}^{t}v_i$, set $P_i(v)=v_i\in V_{\lambda_i}$. These projections satisfy [9]:

(1) $P_i^2=P_i$;
(2) $P_iP_j=0$ for $i\neq j$;
(3) $\sum_{i=1}^{t}P_i=I$.

Now express the new inner product in terms of the original one:

$$\langle x, y\rangle=\sum_{i=1}^{t}(P_ix, P_iy)=\sum_{i=1}^{t}(P_i^*P_ix, y)=\left(\sum_{i=1}^{t}P_i^*P_ix, y\right)=(Tx, y) \tag{15}$$

where $T:=\sum_{i=1}^{t}P_i^*P_i$. Clearly T is Hermitian, and for any $x\in V$,

$$(Tx, x)=\left(\sum_{i=1}^{t}P_i^*P_ix, x\right)=\langle x, x\rangle \tag{16}$$

so T is positive definite [10].

According to the normalization of A and B under the new inner product definition, we have: After converting to the standard inner product, we have $\langle Ax,y\rangle=\langle x,By\rangle$. Translating this into the original inner product gives

$$(TAx, y)=(Tx, By)=\left(B^*Tx, y\right)\Longrightarrow\left(\left(TA\text{-}B^*T\right)x, y\right)=0 \tag{17}$$

Hence $\left(TA\text{-}B^*T\right)x=0$ for all x , i.e. $TA=B^*T$. Therefore

$$TA\text{-}B^*T=0\Longrightarrow TA=B^*T\Longrightarrow B^*=TAT^{-1}\Longrightarrow B=T^{-1}A^*T \tag{18}$$

In the special case where A itself is normal, its eigenspaces V_{λ_i} are mutually orthogonal. For a fixed i and any $x,y\in V$ write , $x=x_i+x_{\perp i}, y=y_i+y_{\perp i}$ with $x_i,y_i\in V_{\lambda_i}$ and $x_{\perp i},y_{\perp i}\in V_{\lambda_i}^{\perp}=\bigoplus_{j\neq i}V_{\lambda_j}$. Then

$$(P_ix, y)=(x_i, y_i+y_{\perp i})=(x_i,y_i)+(x_i, y_{\perp i})=(x_i,y_i) \tag{19}$$

and similarly $(x, P_iy)=(x_i,y_i)=(P_ix, y)$; consequently P_i is Hermitian. Thus

$$T=\sum_{i=1}^{t}P_i^*P_i=\sum_{i=1}^{t}P_i^2=\sum_{i=1}^{t}P_i=I\Longrightarrow B=A^* \tag{20}$$

and (14) reduces to $B=A^*$, which is exactly the classical result for normal operators.

3. Conclusion

This paper has investigated the spectral relationship between a linear operator and its adjoint from a geometric perspective, extending classical results for normal operators to the broader class of diagonalizable operators and providing a unified geometric framework.

First, for normal operators, an independent proof of spectral symmetry was established without relying on the spectral theorem. By using only fundamental properties of inner products, commutativity, orthogonal decomposition, and norm properties, it was shown directly that each eigenvector of a normal operator corresponds to an eigenvector of its adjoint with a conjugate eigenvalue. This approach avoids circular reasoning and offers a more foundational understanding of the classical result.

Second, the notion of spectral symmetry was generalized to arbitrary diagonalizable operators. Based on the decomposition into eigenspaces, a generalized adjoint operator was defined such that it acts by assigning conjugate eigenvalues on each eigenspace. This operator is uniquely determined by the spectral structure and exhibits strong commutativity properties. By introducing a new inner product adapted to the eigenspace decomposition, the eigenspaces become orthogonal under this modified geometry, allowing the original operator and the generalized adjoint to form a pair of mutual adjoints.

Furthermore, a positive self-adjoint transformation was constructed to relate the new inner product to the original one. Revealing a similarity relationship between the generalized adjoint and the standard adjoint. This transformation captures the geometric deviation from orthogonality among eigenspaces. When the eigenspaces are orthogonal, the generalized adjoint coincides with the standard adjoint, showing that normal operators arise as a special case.

Overall, this framework highlights the intrinsic connection between spectral structure and inner product geometry, and provides a new perspective for studying non-normal operators.

References

[1] Conway, J. B. (2000) A Course in Operator Theory. Providence: American Mathematical Society.
[2] Reed, M., & Simon, B. (1980) Methods of Modern Mathematical Physics I: Functional Analysis. 2nd ed. San Diego: Academic Press.
[3] Trefethen, L. N., & Bau, D. (1997) Numerical Linear Algebra. Philadelphia: SIAM.
[4] Kubrusly, C. S. (2012) Spectral Theory of Operators on Hilbert Space. Boston: Birkhäuser.
[5] Hoffman, K., & Kunze, R. (1971) Linear Algebra. 2nd ed. Upper Saddle River: Prentice-Hall.
[6] Axler, S. (2015) Linear Algebra Done Right. 3rd ed. Cham: Springer.
[7] Horn, R. A., & Johnson, C. R. (2013) Matrix Analysis. 2nd ed. Cambridge: Cambridge University Press.
[8] Roman, S. (2008) Advanced Linear Algebra. 3rd ed. New York: Springer.
[9] Halmos, P. R. (1987) Finite-Dimensional Vector Spaces. New York: Springer.
[10] Bhatia, R. (2007) Positive Definite Matrices. Princeton: Princeton University Press.