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Belief Propagation Algorithm Based on Perturbed Message Passing for Solving Constraint Satisfaction Problems

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Abstract. Constraint satisfaction problems usually exhibit a distinct satisfiability transition phenomenon under stochastic models, and an exact phase transition threshold has been strictly proven to exist especially in the RB model. Aiming at the difficulty in solving hard instances of the random binary model with growing domains (RB) near the phase transition region, this paper proposes a guided decimation algorithm based on asynchronous belief propagation with gradual perturbation (P-BP). On the basis of the asynchronous BP-guided decimation framework, the algorithm introduces a linear annealing perturbation mechanism, which enables the variable-to-constraint messages to transit smoothly from deterministic update to the style of Gibbs sampling. Meanwhile, in the late stage of decimation, the variable fixing strategy is changed from greedy selection to direct sampling from the marginal distribution, supplemented by an automatic restart strategy with a maximum of 3 restarts. These improvements retain the asynchronous update, damping factor and A/B/C edge processing rules of the original algorithm, while significantly enhancing the stochastic exploration capability and effectively alleviating the problems of message oscillation, local convergence and error propagation of fixed variables in the phase transition region. Research shows that the combination of gradual perturbation with late-stage probabilistic sampling and a restart mechanism provides an effective way to enhance the robustness of belief propagation-based algorithms for stochastic constraint satisfaction problems.

Keywords: Stochastic Constraint Satisfaction Problems, RB Model, Satisfiability Phase Transition, Belief Propagation, Perturbed Message Passing

1. Introduction

Constraint satisfaction puzzles occupy a crucial status within the cross-disciplinary domain that covers theoretical computer science, discrete mathematics and statistical physics. The random variants of these problems are generally NP-complete tasks and show a prominent satisfiability phase transition once the constraint density gets to a particular critical threshold [1-4]. Carrying out research on stochastic CSPs assists in grasping the complicated performance of large-scale cases and owns broad application value in issues like graph coloring and vertex cover problems [5]. The traditional stochastic CSP models A, B, C and D usually show the feature of trivial unsatisfiability in large-scale situations [6-8]. Acting as an optimized variant of Model B, Model RB rigorously

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confirms the existence of satisfiability phase transition and precisely figures out the phase transition threshold by controlling the quantity of constraints and the dimension of variable domains [9].

As an efficient approximate inference algorithm, belief propagation has been widely used to solve stochastic CSPs [10]. An asynchronous BP-guided decimation algorithm proposed for Model RB improves the success rate of solution search in the phase transition region through asynchronous update, damping factor and non-convergence handling [11]. However, this method is prone to failure on hard instances due to the influence of message oscillation and early error propagation of fixed variables.

To improve the robustness and exploration capability of hard instances in the phase transition region, this paper introduces a gradual perturbation mechanism on the basis of asynchronous BP [12]. A linear annealing parameter γ_perturb is used to realize the smooth transition of messages from deterministic update to the style of Gibbs sampling. In the late stage of decimation, the fixing strategy is changed from greedy arg max to probabilistic sampling from the marginal distribution, supplemented by automatic restarts. These improvements retain the asynchronous update, damping factor and A/B/C edge rules of the original algorithm, while significantly enhancing the stochastic exploration capability.

2. The RB model

A single case belonging to the RB framework is made up of a collection of N variables $X = \{x_1, x_2, \ldots, x_n\}$ alongside a group of $M = rn\ln n$ constraints $C = \{C_1, C_2, \ldots, C_m\}$ (where $r > 0$ is a constant). The variables in the constraints are assigned values selected from the domain $D = \{d_1, d_2, d_3, \ldots, d_N\}$. The scale of the parameter value range is $|D| = d_N$, where $d_N = N^\alpha$ ($\alpha > 0$ is a constant) increases in a polynomial manner along with the total quantity of parameters N . Each constraint C_a is formed by randomly selecting k ($k \geq 2$) different parameters chosen from the total N parameters available. the N variables. With regard to each constraint C_a , we define $Q_a \subset D^k$ as the set of incompatible assignments for these k variables. We select $|Q_a| = p \cdot d_N^k$ assignment combinations in a random fashion without replacement from all d_N^k possible assignment combinations, where $p \in (0{,}1)$ denotes the constraint tightness. Resolving a single case of the RB framework involves locating an allocation scheme (referred to as a resolution outcome) for the N variables that satisfies all M constraints simultaneously. The existence of satisfiability phase transition in the RB model has been strictly proven. Let $Pr\,(SAT)$ stand for the likelihood that a case produced by the RB framework possesses the satisfiability attribute, and subsequently the subsequent conclusion is valid:

Theorem 1 Let $p_s = 1 - \mathrm{e}^{-\frac{\alpha}{r}}$. If $\alpha > \frac{1}{k}, r > 0$ are two constants, and $k\mathrm{e}^{-\frac{\alpha}{r}} \geq 1$, then

$$\lim_{N\to\infty} Pr(\mathrm{SAT})\Big) = \begin{cases} 1, & p < p_s \\ 0, & p > p_s \end{cases} \tag{1}$$

The above theorem indicates that a sharp satisfiability transition occurs in the RB model at the critical value p_s . In addition, almost all instances generated by the RB model at the threshold are difficult to solve.

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3. Belief propagation algorithm based on perturbed message passing

3.1. BP iteration equations

Since the RB model is NP-complete when $k \geq 2$, we take $k = 2$ for discussion in this paper. The factor graph of the binary RB model is shown in Figure 1. In the factor graph, constraint nodes are represented by squares and denoted by $a, b, c, d, \cdots$, while variable nodes are represented by circles and denoted by $i, j, \cdots$. If variable i is included in constraint a, the variable and the constraint are connected by an edge, denoted as (a, i).

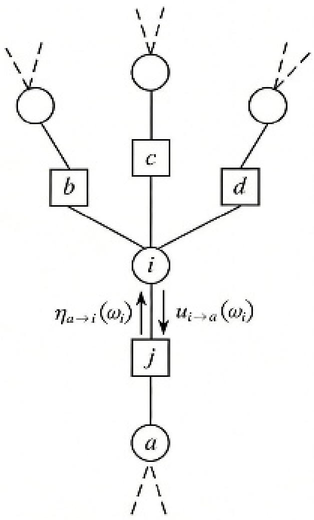

Figure 1. Factor grap

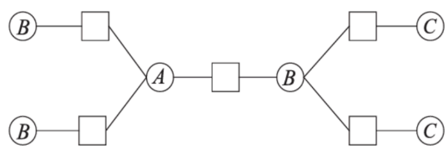

Figure 2. Three different types of variables in the P-BP algorithm

In the operational process of the algorithm implementation, the variables are categorized into three distinct classes as presented in Figure 2: Class A denotes the variables that have already obtained value allocations; Class B signifies the variables that have common constraint connections with the value-allocated variables; Class C indicates the variables that belong to the rest of other situations. For each edge (a, i) in the factor graph structure, two kinds of information applied to belief propagation are stipulated: the first one is the message conveyed from constraint a to variable i, which is marked as $\eta_{a\to i}(\omega_i)$, and this represents the probability that constraint a predicts variable i to choose the value ω_i; the second one is the message delivered from variable i to constraint a, which is denoted as $u_{i\to a}(\omega_i)$, and this stands for the probability that variable i selects the value ω_i without taking the influence of constraint a into account. According to the cavity field theory in the field of statistical physics, the BP iteration calculation equations are demonstrated as follows:

$$u^t_{i\to a}\left(\omega_i\right) = \frac{1}{Z^{i\to a}} \prod\nolimits_{b\in V(i)\backslash a} \eta^t_{b\to i}\left(\omega_i\right) \tag{2}$$

$$\eta^{t+1}_{a\to i}\left(\omega_i\right) = \frac{1}{Z^{a\to i}} \sum\nolimits_{\omega_j\in D, j\in V(a)\backslash i} \delta\left(\omega_i, \omega_j\right) u^t_{j\to a}\left(\omega_j\right) \tag{3}$$

where:

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$$\delta\left(\omega_i, \omega_j\right) = \begin{cases} 0, & (\omega_i, \omega_j) \in Q_a \\ 1, & (\omega_i, \omega_j) \notin Q_a \end{cases}$$

In this context, $Z^{i \to a}$ and $Z^{a \to i}$ are both regarded as normalization coefficients. $V(i) \backslash a$ stands for the collection of adjacent constraints for variable i with the exclusion of constraint a, meanwhile $V(a) \backslash i$ represents the group of adjacent variables for constraint a omitting variable i.

3.2. The P-BP algorithm

This paper makes the following modifications to the BP algorithm in Reference [10], which we call Perturbed Belief Propagation (P-BP algorithm).

In the process of computing the information signals dispatched from variables to constraint conditions, we take into account the impact exerted by the messages in the present iteration step and the former iteration step upon the signals sent from constraint conditions to variables. As a result, prior to the activation of the perturbation mechanism, we utilize:

$$u_{i \to a}^{t+1}(\omega_i) = (1 - \gamma) \cdot u_{i \to a}^{t}(\omega_i) + \gamma \cdot u_{i \to a}^{t+1}(\omega_i) \tag{4}$$

The information signals conveyed from parameters to restriction rules are refreshed subsequent to Equation 2, in which γ serves as a damping coefficient that needs to be optimized and adjusted. In the design of the perturbation mechanism, we choose to introduce Gibbs sampling perturbation to the variable-to-constraint messages $v_{i \to a}$. In the asynchronous update process, for each variable-to-constraint message $u_{i \to a}$, we first calculate its deterministic value u_{bp} according to Formula 2, and then introduce a perturbation mechanism to enhance the exploration capability of the algorithm. Specifically, we define a perturbation intensity parameter $\gamma_{\text{perturb}}(t)$ that is linearly annealed with the number of iterations t:

$$\gamma_{\text{perturb}}\left(t\right) = min\left\{1.0, max\left\{0, \frac{t - t_{\text{start}}}{t_{\text{max}} - t_{\text{start}}}\right\} \times \gamma_{\text{perturb}}^{\text{max}}\right\}, \quad t \geq t_{\text{start}} \tag{5}$$

where t_{start} is the starting iteration of perturbation, t_{max} is the maximum number of iterations, and $\gamma_{\text{perturb}}^{\text{max}}$ is the maximum perturbation intensity. When $\gamma_{\text{perturb}}(t) > 0$, Gibbs sampling is performed from the current marginal distribution u_{bp} with probability $\gamma_{\text{perturb}}(t)$, a value $\omega \sim u_{\text{bp}}$ is sampled, and a soft one-hot vector u_{gibbs} is constructed (the corresponding position is 1, and the others are 0). Then, the perturbed message $u_{\text{perturbed}}$ is calculated as:

$$u_{\text{perturbed}} = (1 - \gamma_{\text{perturb}})u_{\text{bp}} + \gamma_{\text{perturb}} u_{\text{gibbs}} \tag{6}$$

The perturbed message is then smoothed by the original damping factor γ:

$$u_{\text{new}} = (1 - \gamma)u_{\text{perturbed}} + \gamma u_{\text{old}} \tag{7}$$

In the asynchronous update process, for each variable-to-constraint message $u_{i \to a}$, its deterministic value u_{bp} is first calculated. Then, a uniform random number $r \sim U[0,1]$ is

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generated. If $r < \gamma_{\text{perturb}}(t)$, Gibbs sampling is performed from the current marginal distribution v_{bp} . This probability trigger mechanism ensures that the perturbation intensity $\gamma_{\text{perturb}}(t)$ directly controls the occurrence frequency of stochastic perturbation, thus realizing a smooth transition from deterministic BP to stochastic Gibbs sampling. Through the above gradual perturbation mechanism, the algorithm maintains deterministic update in the early stage to ensure fast convergence, and gradually enhances randomness in the late stage to simulate the Gibbs sampling process, thereby effectively alleviating the problems of message oscillation and error propagation of fixed variables in the phase transition region. In the decimation phase, we further optimize the variable fixing strategy. When the perturbation intensity $\gamma_{\text{perturb}}(t) > 0.8$ in the current iteration, the variable fixing method is changed from the greedy $\operatorname{argmax} p(\omega_i)$ of the original algorithm to probabilistic sampling from the marginal distribution:

$$\omega_i \sim p(\omega_i) \tag{8}$$

4. Numerical results

We use the RB model to validate the effectiveness of the proposed P-BP computation approach, where the parameter settings of the algorithm are shown in Table 1. Specifically, we adopt $\alpha = 0.8$ and $r = 3$ to produce 50 RB model instances with sizes $N = \{20,40,60,80\}$.

Table 1. Input parameters of the P-BP algorithm

	N	k	α	r	$t_{\max}$	ε	γ	$\gamma_{\text{perturb}}^{\max}$	t_{start}	max _ restarts
Value	20-100	2	0.8	3	1000	10^{-4}	0.25,0.5,0.75	0.8	300	3

Table 2 lists the corresponding variable domain size N and number of constraints M for various values of d . It can be known from Theorem 1 that the satisfiability phase transition point $p_s \approx 0.23$ when N approaches infinity.

Table 2. Parameters corresponding to different numbers of variables

	$N = 20$	$N = 20$	$N = 20$	$N = 20$
α	0.8	0.8	0.8	0.8
r	3	3	3	3
d_N	11	19	26	33
M	180	443	737	1052

We initially conduct an analysis on the operational performance of the P-BP computation method. The possibility of satisfiability for the fifty manufactured RB model cases is presented in Figure 3. It

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can be noticed that nearly all the cases possess the satisfiability attribute when the constraint strictness p is less than 0.19. The P-BP algorithm is capable of effectively locating a resolution scheme for these cases in the scenario where p is below 0.20. Nonetheless, the achievement possibility of this computation approach turns out to be extremely low when p is greater than 0.22. In spite of this, the P-BP algorithm still demonstrates favorable operational effects when p is near the theoretical satisfiability critical point p_s approximately equal to 0.23, and is able to resolve a tiny quantity of cases when p surpasses the theoretical satisfiability critical point.

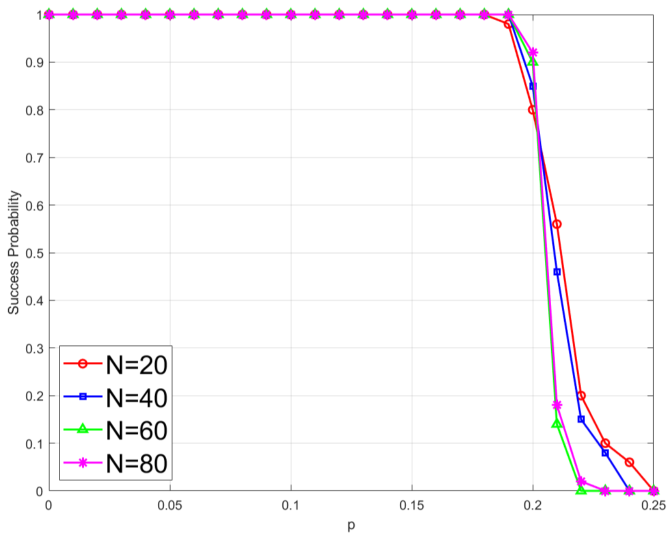

Figure 3. Success probability of the P-BP algorithm (γ= 0.5)

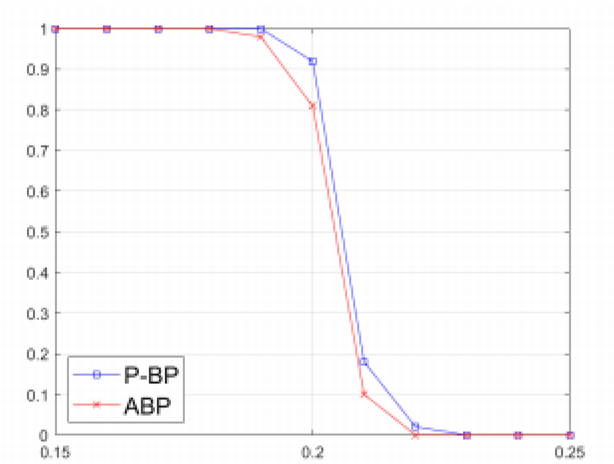

Figure 4. Contrast of the achievement probability among the P-BP approach and the ABP method

We then compare the satisfiability probability of the BP algorithm in Reference [10] and the P-BP algorithm when $\alpha = 0.8, r = 3$ and $N = \{20,40,60,80\}$, as shown in Figure 5. The experimental results show that the P-BP algorithm has stronger convergence than the BP algorithm and improves the efficiency of solution search. By comparing the satisfiability probability with the ABP algorithm ($\gamma = 0.5$) when $N = 80$ (as shown in Figure 4), we find that through the gradual perturbation mechanism and late-stage probabilistic sampling, the P-BP algorithm alleviates the

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problems of falling into local optimum and error propagation in the phase transition region, enhances the stochastic exploration capability, and achieves a slight improvement in efficiency compared with the ABP algorithm.

Ultimately, we carry out an investigation on the execution duration of the P-BP algorithm, and display the runtime of this algorithm under the conditions of $N = 20$ and $\gamma = 0.5$ as a functional relation with p in Figure 6. The execution time of the algorithm grows gradually along with the growth of p , and surges abruptly when p approaches the satisfiability critical point, which represents a typical occurrence for algorithms in the vicinity of the critical threshold.

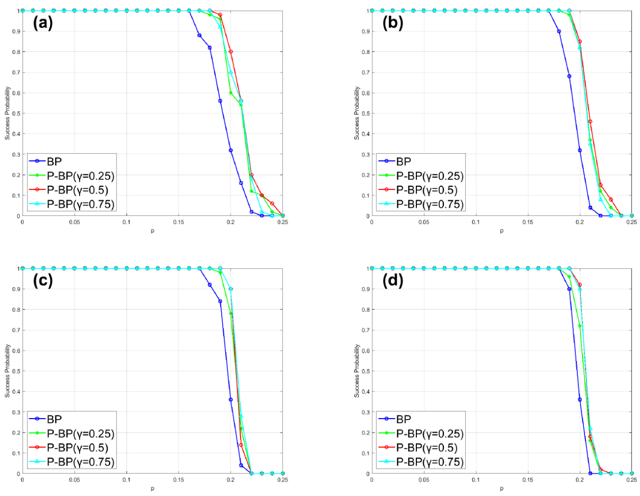

Figure 5. Comparison of the success probability between the P-BP algorithm and the BP algorithm, (a)-(d) represent the cases of$N = \{20,40,60,80\}$ respectively

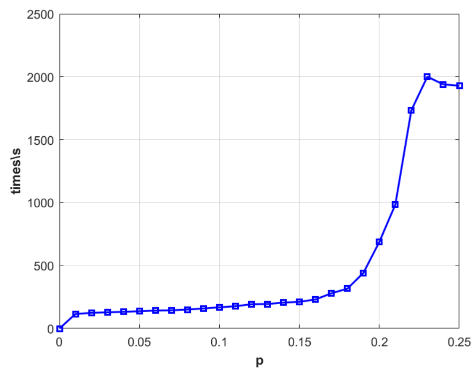

Figure 6. Running time of the P-BP algorithm withN= 20

5. Conclusion

Aiming at the intrinsically hard instances of the growing domain model RB near the phase transition region, this paper proposes a guided decimation algorithm based on asynchronous belief propagation with gradual perturbation (P-BP). By introducing a linear annealing perturbation mechanism (achieving a gradual transition from deterministic update to Gibbs sampling style), a late-stage variable fixing strategy of probabilistic sampling from the marginal distribution, and limited automatic restarts, the algorithm significantly enhances the stochastic exploration capability and robustness on hard instances. Experiments show that near the phase transition region, the P-BP

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algorithm can solve more satisfiable instances on the premise of controllable computational cost compared with the BP algorithm, and exhibits better convergence and overall solution search efficiency.

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