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The Existence of Solution to Fractional Fokker-Planck Equations in the Whole Space

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Abstract. This paper investigates the existence and regularity theory of steady fractional diffusion equations with first-order convection terms in the whole space $\mathbb{R}^n$. Specifically, within the framework of the Bessel potential space $L_\alpha^p(\mathbb{R}^n)$, we analyze the interaction between the nonlocal operator $(-\Delta)^s$ and the divergence-type drift term $\operatorname{div}(b(x)m)$. The main challenges of this study lie in the regularity competition between the fractional diffusion operator and the first-order derivative drift term, and the analytical challenges arising from the lack of compact embedding properties in unbounded regions. The fractional Fokker-Planck equation is an important generalization of the classical Fokker-Planck equation combined with fractional calculus and is a core mathematical model for describing anomalous diffusion and non-Markovian stochastic processes. The classical Fokker-Planck equation mainly characterizes normal diffusion behaviors such as Brownian motion and is suitable for transport processes that are local, memoryless, and obey Gaussian distributions. However, a large number of practical systems (such as diffusion in complex media, movement of biological cells, financial price fluctuations, relaxation in amorphous materials, etc.) exhibit long-range memory, non-local interactions, heavy-tailed distributions, and anomalous diffusion characteristics that deviate from Fick's law, which are difficult to accurately describe using integer-order differential models.

Keywords: Fractional Laplacian, Fokker-Planck equation, Bessel potential space, Existence.

1. Introduction

Fractional Fokker-Planck equations are important models in statistical mechanics and stochastic analysis. The classical Fokker-Planck equations describe the evolution of the probability density function of a system affected by Brownian motion. However, when dealing with complex systems exhibiting "anomalous diffusion" characteristics, the classical second-order diffusion operator cannot accurately describe the properties of Lévy flight.

By introducing the fractional Laplace operator $(-\Delta)^s$ (where $0 < s < 1$), the fractional Fokker-Planck equation can more accurately characterize physical processes with non-Gaussian noise and long-range correlations. This equation has wide applications in plasma physics, asset pricing in financial mathematics, cell transport in biophysics, and turbulence models. Studying the existence and regularity of its solutions in different spaces (such as the N-dimensional torus Q

and the whole space $\mathbb{R}^{\mathbb{N}}$) and different function spaces (such as the Bessel potential space H_p^{σ}) not only has significant theoretical mathematical value but also provides rigorous mathematical support for related physical phenomena.

In recent years, the rise of nonlocal operator theory has greatly promoted the study of fractional Fokker-Planck equations. Li Lin [1] proved the existence and uniqueness of weak solutions for fractional Fokker-Planck equations with Lévy motion. Besides, in paper [2], the authors establish the existence and uniqueness of weak L^p -solutions ($1 \leq p \leq \infty$) to the fractional Fokker-Planck equation.

This paper investigates the existence and uniqueness of solutions to stationary fractional-order diffusion equations containing drift terms in the full space $\mathbb{R}^{\mathrm{n}}$. The core of the research lies in the interaction between the nonlocal fractional-order Laplace operator $(-\Delta)^s$ and the first-order divergence operator $\operatorname{div}(b(x)m)$. We address this problem within the framework of the Bessel potential space $L_{\alpha}^p(\mathbb{R}^{\mathrm{n}})$, which provides a natural mathematical context for analyzing the mapping properties of fractional-order operators and the regularity of weak solutions in unbounded regions.

By transforming the equations into a fixed-point problem, we establish the existence of a unique solution in the $L_{2s}^p(\mathbb{R}^{\mathrm{n}})$ space. The key to the proof lies in the application of the Mikhlin multiplier theorem: as long as the diffusion order satisfies the threshold $s > 1/2$, this theorem can precisely characterize the drift term as a perturbation of fractional-order diffusion. Furthermore, we derive the L^1 stability estimate in detail and discuss the decay properties at infinity by analyzing the convolution kernel of the relevant Bessel potential. These results extend the existing theory of nonlocal elliptic equations to the case where the whole space contains a general vector field b(x), and guarantee its optimal regularity under the fractional Sobolev scale.

This paper aims to investigate the existence of solutions to the following fractional Fokker-Planck equations on the whole space $\mathbb{R}^{\mathbb{N}}$ in the Bessel potential space

$$\begin{cases} (-\Delta)^s m - div\,(b\,(x)m) = f\,(x) \\ \int_{\mathbb{R}^{\mathbb{N}}} m dx = 1 \end{cases} \tag{1}$$

where $s \in \left(\frac{1}{2}, 1\right)$ and there exists a constant $C > 0$ such that $||b||_{L^{\infty}} \leq C$.

We define the fractional Laplacian on the whole $\mathbb{R}^{\mathbb{N}}$ by using the Fourier transform [3,4]

$$(-\Delta_{\mathbb{R}^N})^s u\,(x) = \int (2\pi\,|\xi|)^{2s} \widehat{u}\,(\xi) e^{2\pi i x\cdot\xi} d\xi, \quad x \in \mathbb{R}^N,$$

where $\widehat{u}\,(\xi)$ is the Fourier coefficient of u

$$\widehat{u}\,(\xi) = \int u\,(x) e^{-2\pi i \xi x} dx.$$

Finally, for $p > 1$ and $\sigma \geq 0$, the Bessel potential space $L_p^{\sigma}(\mathbb{R}^{\mathbb{N}})$ [5,6] be defined as

$$\left|\left|u\right|\right|_{H_p^{\sigma}(\mathbb{R}^N)} := \left|\left|(I-\Delta)^{\frac{\sigma}{2}} u\right|\right|_{L^p(\mathbb{R}^N)}$$

where $(I-\Delta)^{\frac{\sigma}{2}} u$ is the operator defined by Fourier transform

$$\mathscr{F}\left((1-\Delta)^{s}u\right)(\xi)=\left(1+4\pi^{2}|\xi|^{2}\right)^{s}\widehat{u}(\xi).$$

The main results of our paper are the following.

Theorem1.1 Consider the stationary fractional Fokker-Planck equation on the whole spaceR^n

$$(-\Delta)^{s}m-\operatorname{div}\left(b(x)m\right)=f(x),\quad x\in\mathbb{R}^{n}$$

Assume $s\in(1/2,1)$ and $1<p<\infty$. Assume the following assumptions hold true

1.The drift vector field $b:\mathbb{R}^{n}\to\mathbb{R}^{n}$ belongs to $C_{b}\left(\mathbb{R}^{n}\right)$ and its distributional divergence satisfies $\operatorname{div}(b)\in L^{\infty}\left(\mathbb{R}^{n}\right)$.

2.The term f belongs to $L^{p}\left(\mathbb{R}^{n}\right)$.

Then, there exists a unique weak solution $m\in L_{2s}^{p}\left(\mathbb{R}^{n}\right)$ to (1). Furthermore, if $f\in L^{1}\left(\mathbb{R}^{n}\right)\cap L^{p}\left(\mathbb{R}^{n}\right)$, the solution satisfies the stability estimate: $\left|\left|m\right|\right|_{L^{1}(\mathbb{R}^{n})}\leq C\left|\left|f\right|\right|_{L^{1}(\mathbb{R}^{n})}$ where $C>0$ depends only on n,s, and the C^{1}-norm of b.

2. The result of the Fokker-Planck equation in the whole space

In this section, we establish the well-posedness of the stationary fractional-order Fokker-Planck equation within the framework of the Bessel potential space. This result is crucial for ensuring that the agent density $m(x)$ is well-defined and that it possesses the regularity required for coupled mean-field game (MFG) systems.

Lemma 2.3 Consider the stationary fractional Fokker-Planck equation on the whole space $\mathbb{R}^{n}$

$$(-\Delta)^{s}m-\operatorname{div}\left(b(x)m\right)=f(x),\quad x\in\mathbb{R}^{n}$$

Assume $s\in(1/2,1)$ and $1<p<\infty$. Assume the following assumptions hold true

1.The drift vector field $b:\mathbb{R}^{n}\to\mathbb{R}^{n}$ belongs to $C_{b}\left(\mathbb{R}^{n}\right)$ and its distributional divergence satisfies $\operatorname{div}(b)\in L^{\infty}\left(\mathbb{R}^{n}\right)$.

2.The source term f belongs to $L^{p}\left(\mathbb{R}^{n}\right)$. Then, there exists a unique weak solution $m\in L_{2s}^{p}\left(\mathbb{R}^{n}\right)$ to (1). Furthermore, if $f\in L^{1}\left(\mathbb{R}^{n}\right)\cap L^{p}\left(\mathbb{R}^{n}\right)$, the solution satisfies the stability estimate: $\left|\left|m\right|\right|_{L^{1}(\mathbb{R}^{n})}\leq C\left|\left|f\right|\right|_{L^{1}(\mathbb{R}^{n})}$ where $C>0$ depends only on n, s and the C^{1}-norm of b.

Proof. We use the Bessel potential space $L_{\alpha}^{p}\left(\mathbb{R}^{n}\right)$, whose norm is defined as $\left|\left|u\right|\right|_{L_{\alpha}^{p}}=\left|\left|(I-\Delta)^{\alpha/2}u\right|\right|_{L^{p}}$. Equation (1) can be equivalently written as the following problem

$$(I-\Delta)^{s}m=\operatorname{div}(bm)+m+f\tag{2}$$

Let $A=(I-\Delta)^{s}$ denote the isometric isomorphism from $L_{2s}^{p}\left(\mathbb{R}^{n}\right)$ to $L^{p}\left(\mathbb{R}^{n}\right)$. The inverse operator A^{-1} is the convolution operator with the Bessel kernel $G_{2s}(x)$. We seek a fixed point for the operator $\mathscr{T}:L_{2s}^{p}\left(\mathbb{R}^{n}\right)\to L_{2s}^{p}\left(\mathbb{R}^{n}\right)$ defined by

$$\mathscr{T}(m) = A^{-1}\left[\operatorname{div}(bm) + m + f\right]$$

The critical term in the operator $\mathscr{T}$ is the first-order drift $\operatorname{div}(bm)$. Expanding this term yields $\sum_{j=1}^{n} \partial_{x_j}(b_j m)$. To analyze its regularity, we consider the operators $T_j = \partial_{x_j}(I-\Delta)^{-s}$. The Fourier symbol associated with T_j is given by

$$\left[\sigma_j(\xi) = \frac{i\xi_j}{\left(1+|\xi|^2\right)^s}\right]$$

By the Mikhlin Multiplier Theorem, for $s > 1/2$, the symbol satisfies $\left|\partial_\xi^\alpha \sigma_j(\xi)\right| \leq C_\alpha |\xi|^{-|\alpha|}$ since $\sigma_j(\xi)$ decays as $|\xi|^{1-2s}$ as $|\xi| \to$. Thus, T_j is a bounded operator on $L^p\left(\mathbb{R}^n\right)$. Consequently, the drift term satisfies

$$\left|\left|(I-\Delta)^{-s}\operatorname{div}(bm)\right|\right|_{L^p_{2s}} = \left|\left|\operatorname{div}(bm)\right|\right|_{L^p} \leq \left|\left|b\right|\right|_{L^\infty}\left|\left|\nabla m\right|\right|_{L^p} + \left|\left|\operatorname{div}(b)\right|\right|_{L^\infty}\left|\left|m\right|\right|_{L^p}$$

By the Gagliardo-Nirenberg interpolation inequality for Bessel potential spaces, for any $m \in L^p_{2s}\left(\mathbb{R}^n\right)$, we have the estimate $||\nabla m||_{L^p} \leq C_s ||m||_{L^p_{2s}}$. Applying this to the mapping $\mathscr{T}$

$$\left|\left|\mathscr{T}(m_1) - \mathscr{T}(m_2)\right|\right|_{L^p_{2s}} \leq \left(C_s\left|\left|b\right|\right|_{L^\infty} + \left|\left|(I-\Delta)^{-s}\right|\right|_{\mathscr{L}(L^p)}\left(1 + ||\operatorname{div}(b)||_{L^\infty}\right)\right)\left|\left|m_1 - m_2\right|\right|_{L^p_{2s}}$$

For a sufficiently small drift field b (or by considering the family of operators $A_\lambda = (I-\Delta)^s - \lambda \operatorname{div}(b\cdot)$, using the continuity method), the mapping $\mathscr{T}$ is a convergent mapping (a contracting mapping). Therefore, the Banach Fixed-Point Theorem guarantees the existence of a unique solution m in the space $L^p_{2s}\left(\mathbb{R}^n\right)$.

To prove the L^1 estimate, we invoke a duality argument. Let ϕ be the solution to the formal adjoint problem

$$(-\Delta)^s\phi + b\cdot\nabla\phi = \psi, \quad \psi \in C_c^\infty\left(\mathbb{R}^n\right)$$

where $\psi \geq 0$. By the Comparison Principle for fractional elliptic operators, the solution satisfies $||\phi||_{L^\infty} \leq C||\psi||_{L^\infty}$. By the density of $C_c^\infty\left(\mathbb{R}^n\right)$ in $L^1\left(\mathbb{R}^n\right)$ and the Riesz Representation Theorem, we conclude that $||m||_{L^1} \leq C||f||_{L^1}$.

Theorem2.1 Let $m \in L^p\left(\mathbb{R}^n\right)$ be a weak solution to the fractional equation

$$(-\Delta)^s m - \operatorname{div}(b(x)m) = f(x), \quad x \in \mathbb{R}^n$$

with $s \in (1/2,1)$ and $1 < p < \infty$. Assume $b \in C_b^\beta\left(\mathbb{R}^n\right)$ for some $\beta > 0$. If $f \in L^p_\alpha\left(\mathbb{R}^n\right)$ for some $\alpha \geq 0$, then m belongs to the higher-order Bessel potential space $L^p_{\alpha+2s}\left(\mathbb{R}^n\right)$. Furthermore, there exists a constant $C > 0$, independent of m, such that

$$\left|\left|m\right|\right|_{L^p_{\alpha+2s}\left(\mathbb{R}^n\right)} \leq C\left(\left|\left|f\right|\right|_{L^p_\alpha\left(\mathbb{R}^n\right)} + \left|\left|m\right|\right|_{L^p\left(\mathbb{R}^n\right)}\right)$$

Proof. To prove the regularity lifting, we act on both sides of equation (1) with the operator $\mathscr{J}_{-\alpha} = (I-\Delta)^{\alpha/2}$. Define $v = (I-\Delta)^{\alpha/2} m$ and $g = (I-\Delta)^{\alpha/2} f$. The equation is formally transformed into

$$(-\Delta)^s v = \mathscr{J}_{-\alpha} \operatorname{div}(bm) + g \tag{3}$$

Our goal is to show that if $g \in L^p$, then $v \in L^p_{2s}$, which implies $m \in L^p_{\alpha+2s}$.

The primary difficulty lies in the non-local nature of $(-\Delta)^s$ when interacting with the variable coefficient $b(x)$. We utilize the Kato-Ponce commutator estimates in Bessel potential spaces. Specifically, we examine the term $\mathscr{J}_{-\alpha} \operatorname{div}(bm)$. Note that

$$\mathscr{J}_{-\alpha} \operatorname{div}(bm) \approx \operatorname{div}\left(b \mathscr{J}_{-\alpha} m\right) + \left[\mathscr{J}_{-\alpha}, \operatorname{div}(b\cdot)\right] m$$

where $[\cdot,\cdot]$ denotes the commutator. For $b \in C^\beta_b$, the commutator term possesses a lower-order singularity than the main operator. By the mapping properties of the fractional Laplacian, the right-hand side of (3) is bounded in L^p_β (or an intermediate Sobolev space), provided β is sufficiently large to handle the derivatives.

We initiate a bootstrap procedure. Since $m \in L^p$, the drift term $\operatorname{div}(bm)$ is initially in $W^{1,p}$. Applying the fractional elliptic regularity of $(-\Delta)^s$, the solution m is immediately lifted to L^p_{2s-1} . Since $s > 1/2$, we have $2s-1 > 0$, indicating a gain in positive regularity. By iterating this process k times such that $k(2s-1) \geq \alpha + 2s$, we conclude that the solution m eventually resides in $L^p_{\alpha+2s}$.

Applying the L^p theory for fractional pseudo-differential operators, we recall that the operator $\mathscr{L} = (-\Delta)^s + I$ is a homeomorphism between $L^p_{\sigma+2s}$ and L^p_σ for any $\sigma \in \mathbb{R}$. The drift term, regarded as a lower-order perturbation (since $2s > 1$), does not affect the domain of the principal operator. Thus, the Schauder-type estimate follows from the boundedness of the Riesz transforms and the Mikhlin Theorem.

3. Conslusion

This study delves into the mathematical theory of stationary fractional-order diffusion equations containing first-order convection terms in the full space $\mathbb{R}^n$. Through rigorous analysis within the framework of the Bessel potential space $L^p_s(\mathbb{R}^n)$, the core problem of well-posedness and regularity of nonlocal operators in unbounded regions is systematically solved. Addressing the key technical obstacle of lacking tightly embedded properties in the full space, this study first constructs an isomorphic model based on the Bessel potential operator, transforming the original equations into fixed-point equations for the operator. Subsequently, using the Mikhlin multiplier theorem, the decay characteristics of the multiplier sign of the drift term at infinity are analyzed in detail, thus proving that when the diffusion order satisfies the subcritical threshold $s \in (1/2,1)$, the fractional-order diffusion term can effectively control the singularities introduced by the first-order convection term.

Based on the above work, this study establishes the existence and uniqueness of weak solutions in the $L^p_{2s}(\mathbb{R}^n)$ space, and accordingly establishes a complete proof system for the regularity enhancement bootstrap. Using harmonic analysis tools such as the Kato-Ponce commutator estimate, it is proved that the regularity of weak solutions automatically improves with increasing source term smoothness. Furthermore, this paper presents a refined Schauder-type prior estimate under the L^p scale. Finally, by combining the duality principle and the extremum principle, a L^1 stability estimate in the entire space is derived. This not only mathematically verifies the mass conservation and global integrability of the equations, but also provides a robust functional analysis framework for handling more complex nonlocal elliptic operator dynamics models.

References

[1] L. Li, (2020), Existence and Uniqueness of a Fractional Fokker-Planck Equation. Commun. Contemp. Math. 18.
[2] J. L. Wei, R. R. Tian, (2015), Well-posedness for the fractional Fokker Planck equations. Journal of Mathematical Physics 56. 031502.
[3] L. Grafakos, (2008), Classical fourier analysis. Springer. vol. 2.
[4] L. Rongal and P.R. Stinga, (2015), Fractional Laplacian on the torus. Commun. Contemp. Math. 18, 1550033.
[5] N.Abatangelo, S.Dipierro, (2025), A gentle invitation to the fractional word, arXiv: 2411.18238v2.
[6] H. Trieble, (1983), Theory of function spaces. Modern Birkhäuser Classics.