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The Existence of Classical Solutions to Fractional Mean Field Games

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Abstract. This paper investigates the existence and properties of solutions to a class of stationary fractional mean-field games. The system is coupled with a Hamilton Jacobi Bellman equation with a fractional Laplace operator and a steady fractional Fokker Planck equation describing the agent distribution. Compared to traditional second order diffusion models, the fractional dynamics considered in this paper better characterize stochastic processes with anomalous diffusion properties. The authors explore the normality, uniqueness, and asymptotic behavior of the agent distribution density of the system solutions under different nonlocal diffusion orders $s \in (1/2,1)$. Using the variational method or fixed-point theory, it is proven that the stationary system possesses classical or weak solutions under specific monotonicity or growth conditions of the coupled terms. Furthermore, the paper analyzes the impact of the nonlocality of the fractional operator on the game equilibrium state. The results not only generalize classical mean-field game theory but also provide theoretical support for the application of nonlocal diffusion models in economics and social sciences.

Keywords: Stationary mean field games, Hamilton-Jacobi equation, Fractional, Kolmogorov Fokker Planck equation

1. Introduction

Mean Field Games was proposed by Lasry and Lions [1] in 2007, which is a mathematical theory to study stochastic differential games with a large number of interchangeable players.

The Mean Field Games models is composed of Hamilton Jacobi equation and Kolmogorov Fokker Planck equation, stationary focusing mean field games systems with different assumptions [2,3] have been studied.

This paper is investigated the system below on the N -dimensional torus $T \coloneqq \mathbb{R}^N/\mathbb{Z}^N$. Seeking to obtain a real constant $\lambda \in \mathbb{R}$ and (u,m) solving

$$\begin{cases} (-\Delta)^s u + \frac{1}{\gamma}|\nabla u|^\gamma + \lambda = m^{q-1} + V\left(x\right) \\ (-\Delta)^s u - div\left(m|\nabla u|^{\gamma-1}\right) = 0, \\ \int mdx = 1 \end{cases} \tag{1}$$

Here for $s \in \left(\frac{1}{2}, 1\right)$ and $u : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$, the fractional Laplacian is defined by using the Fourier transform [4,5]

$$(-\Delta_T)^s u(x) = (2\pi)^{2s} \sum_{k\in\mathbb{Z}^{\mathbb{N}}} |k|^{2s} \widehat{u}(k) e^{2\pi i k\cdot x},$$

where $\widehat{u}(k)$ is the Fourier coefficient of u :

$$\widehat{u}(k) = \int_T u(x) e^{-2\pi i k x} dx.$$

Firstly, assume the Hamiltonian $H(p) = \frac{1}{\gamma}|p|^{\gamma} : \ \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is strictly convex and also locally Lipschitz continuous.

And for the $f(x, m) = m^{q-1}$

$$1 < q < 1 + \frac{(2s-1)}{N} \frac{\gamma}{\gamma-1}, \ \gamma' > \frac{N}{2s-1}, \tag{2}$$

in which $\gamma' = \frac{\gamma}{\gamma-1}$.

Moreover, potential V satisfies for $C_V > 0$

$$0 < V(x) < C_V\left(1 + \frac{1}{|x|}\right), \ x \in \mathbb{R}^{\mathbb{N}}. \tag{3}$$

Set

$$K = \left\{ \left(m, w\right) \in L^1\left(T\right) \cap H_{\gamma'}^{2s-1}\left(T\right) \times L^{\gamma'}\left(T\right) \right.$$

$$s.t. \int_T m(-\Delta)^s \varphi dx = \int_T w \cdot \nabla\varphi dx, \ \forall\varphi \in C^{\infty}(T),$$

$$\left. \int m \, dx = 1, \ m \geq 0 \, a.e. \right\}$$

Associate (1) with the energy

$$F\left(m, w\right) = \begin{cases} \int m \, L\left(-\frac{w}{m}\right) + V\left(x\right)m + F\left(x, m\right) dx, & \left(m, w\right) \in E \\ +\infty & otherwise \end{cases} \tag{4}$$

where

$$L\left(-\frac{w}{m}\right) = \begin{cases} \frac{1}{\gamma'}\left|\frac{w}{m}\right|^{\gamma'} & if \, m > 0, \\ 0 & if m = 0, w = 0 \\ +\infty & otherwise \end{cases} \tag{5}$$

$$F\left(x,m\right) \coloneqq \begin{cases} \int_0^m f(x,n)dn & \left(m,w\right) \in \ E \\ +\infty, & otherwise \end{cases} \tag{6}$$

Finally, for $p > 1$ and $\sigma \geq 0$, $H_p^\sigma(T)$[6] be defined as

$$H_p^\sigma(T) \coloneqq \left\{u \in L^p(T) : (I-\Delta)^{\frac{\sigma}{2}} u \in L^p(T)\right\}$$

$$with \ \ ||u|\Big|_{H_p^\sigma(T)} \coloneqq \Big|\Big|(I-\Delta)^{\frac{\sigma}{2}} u||_{L^p(T)},$$

and

$$||u\Big|\Big|_{H_p^\sigma(T)} = ||u||_{L^p(T)} + \Big|\Big|(\text{-}\Delta)^{\frac{\sigma}{2}} u||_{L^p(T)}.$$

where $(I-\Delta)^{\frac{\sigma}{2}} u$ is the operator defined by multiple Fourier series

$$\left[(I-\Delta)^{\frac{\sigma}{2}} u(x) = \sum_{k\in\mathbb{Z}^{\mathbb{N}}} \left(1+4\pi^2|k|^2\right)^{\frac{\sigma}{2}} \widehat{u}(k) e^{2\pi i k\cdot x}.\right]$$

The result of our paper is stated below.

Theorem1.1 Under the hypotheses (2) and (3), for $p > 1$ and $\theta < 2s-1$, system (1) is shown to have a classical solution (u,λ,m) belongs to $C^{2s+\theta}(Q) \times \mathbb{R} \times H_p^{2s-1}(Q)$

The structure of this paper is as follows. Section 2 is devoted to presenting the existence, uniqueness and a-priori estimates of solutions; in Section 3 we investigate several existence and regularity results concerning Hamilton–Jacobi equations. Section 4 proves the existence of Mean Field Games system.

2. Some results of the Fokker-Planck equation

Now we provide some results of fractional Fokker-Planck equations.

There are continuous embeddings for $H_p^\sigma(T)$.

Lemma 2.1(1) For $s > 0, p > 1$, $q > 0$, $H_p^{s+q}(T) \subseteq W_p^s(T) \subseteq H_p^{s-q}(T)$.

(2) If $ps > d$ and $s - \frac{d}{p}$ is not an integer, then $H_p^s(T) \subseteq C^{s-\frac{d}{p}}(T)$.

(3) Assume that $\frac{1}{p} - \frac{s}{n} < \frac{1}{q} \leq \frac{1}{p} < 1$, then $H_p^s(T) \subseteq L^q(T)$.

We give two standard results that will be used.

Proposition 2.1 (1) For every smooth f, g, the following inequality holds for any $s \in (0,1)$

$$\int (-\varDelta)^{s-\frac{1}{2}} f (-\varDelta)^{\frac{1}{2}} g dx = \int f (-\varDelta)^s g dx = \int (-\varDelta)^s f g dx. \tag{7}$$

(2) Let $\phi \in C^\infty(T)$, then we have a constant $C_1 > 0$ such that

$$||\nabla \phi||_{L^p(T)} \le C_1 \left|\left|(-\Delta)^{\frac{1}{2}} \phi\right|\right|_{L^p(T)}. \tag{8}$$

Proof. (1) The functions f and g can be written by multiple Fourier series

$$f(x) = \sum_{l \in \mathbb{Z}^{\mathbb{N}}} \widehat{f}(l) e^{2\pi i l \cdot x} \qquad \text{and} \qquad g(x) = \sum_{l \in \mathbb{Z}^{\mathbb{N}}} \hat{g}(l) e^{2\pi i l \cdot x}$$

Then

$$\begin{aligned}\int_T (-\triangle)^{s-\frac{1}{2}} f(-\triangle)^{\frac{1}{2}} g dx &= \int_T (2\pi)^{2s-1} \sum_{k \in \mathbb{Z}^h} |k|^{2s-1} \widehat{f}(k) e^{2\pi i k \cdot x} \cdot (2\pi) \sum_{l \in \mathbb{Z}^h} |l| \hat{g}(l) e^{2\pi i l \cdot x} dx \\ &= (2\pi)^{2s} \sum_{k,l \in \mathbb{Z}^N} |k|^{2s-1} |l| \widehat{f}(k) \hat{g}(l) \int_T e^{2\pi i (k+l) \cdot x} dx \\ = (2\pi)^{2s} \int_T \sum_{k,l \in \mathbb{Z}^N} \widehat{f}(k) e^{2\pi i k \cdot x} |l|^{2s} \hat{g}(l) \int_T e^{2\pi i l \cdot x} dx &= \int_T f(-\Delta)^s g dx \\ = (2\pi)^{2s} \int_T \sum_{k,l \in \mathbb{Z}^N} |k|^{2s} \widehat{f}(k) e^{2\pi i k \cdot x} \hat{g}(l) e^{2\pi i l \cdot x} dx &= \int_T (-\Delta)^s f g dx.\end{aligned}$$

Lemma 2.2 For any $\phi \in H_p^{\sigma}(T)$, and $\sigma \ge 0$. The solution $m \in H_p^{2s-1+\sigma}(T)$ to the equation

$$(-\Delta)^s m = div(\phi), \quad with \ \int m dx = 1. \tag{9}$$

can be found. Furthermore, a constant $C > 0$ can be found, which depends on p satisfies

$$\left|\left|m\right|\right|_{H_p^{2s-1+\sigma}(T)} \le C ||\phi||_{H_p^{\sigma}(T)}. \tag{10}$$

Proof. We begin by analyzing the problem

$$-\Delta u = \operatorname{div}(\phi), \quad \text{with} \quad \int_T u dx = 0. \tag{11}$$

has a unique solution $u \in H_p^{1+\sigma}(T)$.

Suppose ϕ is smooth, and u is a solution to (12) and $v \in C^\infty(T)$. Then multiplying equation (12) by $(-\Delta)^{\frac{\sigma}{2}} v$, using (8) and (9), then we obtain that

$$\int_T u(-\Delta)^{1+\frac{\sigma}{2}} v dx \le C ||\phi||_{H_p^{\sigma}(T)} \left|\left|(-\Delta)^{\frac{1}{2}} v\right|\right|_{L^{p'}(T)},$$

where $p' = \frac{p}{p-1}$. By selecting an arbitrary test function $\psi = (-\triangle)^{\frac{1}{2}} v$, which satisfies the condition of having zero average, we get

$$\int_T u(-\Delta)^{\frac{1+\sigma}{2}} \psi dx \le C \left|\left|\phi\right|\right|_{H_p^{\sigma}(T)} ||\psi||_{L^{p'}(T)}, \quad \forall \psi \in C^\infty(T) \text{ with } \int_T \psi = 0,$$

so

$$\left|\left|u\right|\right|_{H_p^{1+\sigma}(T)} \le C||\phi||_{H_p^{\sigma}(T)}. \tag{12}$$

Let $m := 1 + (-\Delta)^{1-s}u \in H_p^{2s-1+\sigma}(T)$, and $(-\Delta)^s m = -\Delta u$, we get $\int_T m dx = 1$ and m constitutes the solution to (10). We then have

$$||m||_{H_p^{2s-1+\sigma}(T)} \le ||u||_{H_p^{\sigma-1+2s}(T)} + ||u||_{H_p^{\sigma+1}(T)}. \tag{13}$$

Notice that $s \in \left(\frac{1}{2}, 1\right)$, and

$$\left[H_p^{\sigma+1} \subset H_p^{\sigma-1+2s}, \left|\left|u\right|\right|_{H_p^{\sigma-1+2s}(T)} \le C\left|\left|u\right|\right|_{H_p^{\sigma+1}(T)}.\right.$$

Hence, (14) together with (13), gives (11).

Lemma 2.3 Suppose $q > 1$, $m \in L^1$ (T), then $\int_T m dx = 1$ and

$$\int_T m(-\Delta)^s \phi \le C||\nabla\phi||_{L^{q'}(T)}, \quad \forall\phi \in C^1(T), \tag{14}$$

with $q' = \frac{q}{q-1}$, for $C' > 0$. So $(-\Delta)^{s-\frac{1}{2}} m \in L^q(T)$ and

$$\left|\left|(-\Delta)^{s-\frac{1}{2}} m\right|\right|_{L^q(T)} \le C'. \tag{15}$$

Proof. Set $m_\varepsilon := m \star \chi_\varepsilon$, then (15)

$$\int_T m_\varepsilon(-\Delta)^s \phi \le C'\left|\left|\nabla\phi\right|\right|_{L^{r'}(T)}, \quad \forall\phi \in C^1(T).$$

So, due to (8) and (9), we get

$$\int_T (-\Delta)^{s-\frac{1}{2}} m_\varepsilon (-\Delta)^{\frac{1}{2}} \phi dx \le C'\left|\left|(-\Delta)^{\frac{1}{2}}\phi\right|\right|_{L^{r'}(T)},$$

which yields the desired estimate (16) for m_ε. The case of m is then obtained by sending $\varepsilon \to 0$.

3. Some priori estimates of the Hamilton-Jacobi equation

In this section, we investigate the Hamilton–Jacobi equation

$$(-\Delta)^s u + \frac{1}{\gamma}|\nabla u|^\gamma + \lambda = m^{q-1} + V\left(x\right) \tag{16}$$

where $m^{q-1} + V(x)$ is locally Lipschitz continuous. We have the following results [2, Theorem 3.1].

Theorem 3.1. 1. Let u be a continous periodic solution to equation (17), we can then find $G > 0$ ($\left|\left|u\right|\right|_{L^\infty(T)}$, $\left|\lambda\right|$, $\left|\left|f\right|\right|_{L^\infty(T)}$), satisfies $\left|\left|\nabla u\right|\right|_{L^\infty(T)} \le G$. Furthermore, we can establish

$K > 0$, (u , $\left|\left|f\right|\right|_{L^\infty(T)}$), for which $\left|\left|u\right|\right|_{L^\infty(T)} \leq K$

2. The $\lambda \in R$ exists ensuring (17) admits $u \in W^{1,\infty}(T)$ with

$$\lambda = \sup\{c \in R\, s.t.\ exists\ u \in W^{1,\infty}(T)\ s.t.\ (-\Delta)^s u + \frac{1}{\gamma}|\nabla u|^\gamma + \lambda \leq m^{q-1} + V\left(x\right) \quad (17)$$

In addition, if $m^{q-1} + V(x) \in C^\theta(T)$, for $\theta \in (0{,}1)$, then $u \in C^{2s+\alpha}(T) \cup H_p^{2s}(T)$, for all $\alpha < \theta$ and $p > 1$.

4. Existence

This present section is devoted to proving the existence of classical solutions to (1).

Proposition 4.1 Suppose $(m, w) \in K$ satisfies for some $Q > 0$

$$\int m \left|\frac{w}{m}\right|^{\gamma'} \leq Q.$$

Deduce the existence of $\delta > 0$ and $C > 0$ satisfying

$$||m||_{L^q}^{q(1+\delta)} \leq \int m \left|\frac{w}{m}\right|^{\gamma'} \leq CQ \tag{18}$$

where q is as in (2).

In addition, for all $\alpha \in \left(0{,}2s - 1 - \frac{N}{\gamma'}\right), C > 0$ can be found, which depends on $\backslash alpha$, satisfies

$$||m||_{C^\alpha} \leq \int m \left|\frac{w}{m}\right|^{\gamma'} \leq CQ \tag{19}$$

Proof. Due to $(m, w) \in K$, $H_{\gamma'}^{2s-1}(T)$ and γ' satisfies (2), we get $m \in C_\alpha(T)$, for every $\alpha \in \left(0{,}2s - 1 - \frac{N}{\gamma'}\right)$.

Set $p > 1$ and define r_p as the following

$$\frac{1}{r_p} = \frac{1}{\gamma'} + \left(1 - \frac{1}{\gamma'}\right)\frac{1}{p}. \tag{20}$$

It follows that $r_p < \min\{p, \gamma'\}$.

Using (5) and Hölder inequality, we have that

$$\int_T m(-\Delta)^s \phi dx \leq \left(\int_T m\left|\frac{w}{m}\right|^{\gamma'} dx\right)^{\frac{1}{\gamma'}} |m|_{L^p(T)}^{\frac{1}{\gamma}} |\nabla\phi|_{L^{r_p'}(T)}$$

for any $\phi \in C^\infty(T)$. We have $\frac{1}{\gamma'} + \frac{1}{\gamma p} + \frac{1}{r_p'} = 1$ by (21) and $r_p' = \frac{r_p}{r_p - 1}$.

Therefore, by Lemma 2.3 we obtain that

$$\left|\left|(-\Delta)^{s-\frac{1}{2}}m\right|\right|_{L^{r_p}(T)} \leq C\left(\int_T m\left|\frac{w}{m}\right|^{\gamma'}dx\right)^{\frac{1}{\gamma'}}||m|\Big|^{\frac{1}{\gamma}}_{L^p(T)}, \tag{21}$$

for $C > 0$. Futhermore, we have

$$\Big|\,||m||_{L^{r_p}(T)} \leq ||m|\Big|^{\frac{1}{\gamma}}_{L^p(T)}||m|\Big|^{\frac{1}{\gamma'}}_{L^1(T)} = ||m||^{\frac{1}{\gamma}}_{L^p(T)}. \tag{22}$$

By (22) and (23), we get that

$$\begin{aligned} ||m||_{H^{2S-1}_{rp}(T)} &= ||m||_{L^{r_p}(T)} + ||(-\Delta)^{s-\frac{1}{2}}m||_{L^{r_p}(T)} \\ &\leq C\left(\left(\int_T m|\frac{w}{m}|^{\gamma'}\mathrm{d}x\right)^{\frac{1}{\gamma'}} + 1\right)||m||^{\frac{1}{\gamma}}_{L^q(T)} \end{aligned} \tag{23}$$

Now we prove (19). Let $r = r_q$ in (21) and we set $p = q$. Suppose $r^\star$ is the following

$$\begin{cases} \frac{1}{r^*} = \frac{1}{r} - \frac{2s-1}{N} \\ \quad\quad \infty \end{cases} \tag{24}$$

Notice that by (2), we can see that

$$q < r^\star. \tag{25}$$

So, we have $C > 0$ such that

$$\Big|\,||m||_{L^q(T)} \leq C||m||_{H^{2s-1}_r(T)}.$$

Therefore, we obtain

$$\Big|\,||m||_{L^q(T)} \leq C||m|\Big|_{H^{2s-1}_r(T)}$$

$$\leq C\left(\left(\int_T m\left|\frac{w}{m}\right|^{\gamma'}\mathrm{d}x\right)^{\frac{1}{\gamma'}} + 1\right)||m|\Big|^{\frac{1}{\gamma}}_{L^q(T)},$$

$$\Big|\,|m|\Big|_{H^{2s-1}_r(T)} \leq C\left(\left(\int_T m|\frac{w}{m}|^{\gamma'}\mathrm{d}x\right) + 1\right).$$

And due to (26), using interpolation and by (24), we have

$$\Big|\,||m||_{L^q(T)} \leq ||m||^{\theta}_{L^*(T)}||m||^{1-\theta}_{L^1(T)}$$

$$\leq ||m||^{\theta}_{H_p^{2s-1}(T)} ||m||^{1-\theta}_{L^1(T)}$$

$$\leq C\left(\left(\int_T m\left|\frac{w}{m}\right|^{\gamma'}\mathrm{d}x\right)^{\frac{\theta}{\gamma'}}+1\right)||m|\Big|^{\frac{\theta}{\gamma}}_{L^q(T)}$$

where θ is such that

$$\tfrac{1}{q} = 1-\theta+\tfrac{\theta}{r^{\star}}. \tag{26}$$

Then substituting (21) and (25) into (27), we check that

$$\tfrac{1}{\theta} = 1-\tfrac{1}{\gamma'}+\tfrac{2s-1}{N}\,\tfrac{q}{q-1}. \tag{27}$$

We get that

$$\Big||m|\Big|^{\left(1-\frac{\theta}{\gamma}\right)\frac{\gamma'}{\theta}}_{L^q(T)} \leq C\left(1+\int_T m\left|\tfrac{w}{m}\right|^{\gamma'}\mathrm{d}x\right). \tag{28}$$

Using (28) we check that

$$\left(1-\tfrac{\theta}{\gamma}\right)\tfrac{\gamma'}{\theta} = \gamma'\tfrac{2s-1}{N}\,\tfrac{q}{q-1} = (1+\delta)q,$$

where in virtue of $1<q<1+\frac{(2s-1)}{N}\,\frac{\gamma}{\gamma-1}$,
we get that

$$\delta = \tfrac{1}{q-1}\left(\tfrac{\gamma'(2s-1)+N}{N}-q\right) > 0.$$

Therefore this and (20) imply (19).
Then we prove (20). By (2), let p in (21) such that

$$\tfrac{N}{2s-1} < r_p < \gamma'.$$

Therefore, from (24)

$$\Big||m|\Big|_{H_{r_p}^{2s-1}(T)} \leq C\left(\left(\int_T m\left|\tfrac{w}{m}\right|^{\gamma'}\mathrm{d}x\right)+1\right) \tag{29}$$

$$||m||_{L^p} \leq C\left(\left(\int_T m\left|\tfrac{w}{m}\right|^{\gamma'}\mathrm{d}x\right)+1\right)$$

From the second estimate in (30) and $\alpha \in \left(0{,}2s-1-\frac{N}{\gamma'}\right)$, $H_{r_p}^{2s-1}(T) \subseteq C^{\alpha}(T)$, then we get

$$\left||m|\right|_{C^{\alpha}(T)} \leq C \int_T m\left|\frac{w}{m}\right|^{\gamma'} dx \leq CQ.$$

Lemma 4.1 For any $(m,w) \in \mathscr{E}$, we can find a constant C, depending on $N,\gamma,q,$ Satisfying

$$\mathscr{F}(m,w) \geq C.$$

Specifically, there holds

$$e = \inf_{(m,w)\in\mathscr{E}} \mathscr{F}(m,w).$$

Furthermore, there exists $(m,w) \in \mathscr{E}$, satisfying

$$\mathscr{F}(m,w) = \min_{(m,w)\in\in\mathscr{K}} \mathscr{F}(m,w).$$

Proof. Recalling that the definition of $\mathscr{F}(m,w)$ and (19), we get

$$F(m,w) = \int mL\left(-\frac{w}{m}\right) + V(x)m + F(x,m)dx$$

$$\geq C||m||_{L^q(T)}^{q(1+\delta)} + \frac{1}{q}\,||m||_{L^q(T)}^{q} \geq C.$$

So there exists $e = \inf_{(m,w)\in\mathscr{E}} \mathscr{F}.$

Let us consider the minimizing sequence (m_n, w_n). Therefore $\mathscr{F}(m_n,w_n) \leq e+1,$ for every n large enough.

Therefore, again by (19), we get

$$\frac{1}{\gamma'} \int m_n\left|\frac{w_n}{m_n}\right|^{\gamma'} \mathrm{d}x \ \leq e+1+\frac{1}{q}\int_T m_n^q(x)dx$$

$$\leq e+1+C'\left(\int m_n\left|\frac{w_n}{m_n}\right|^{\gamma'} \mathrm{d}x\right)^{\frac{1}{1+\delta}}.$$

This implies that the quantity $\int_T m\left|\frac{w}{m}\right|^{\gamma'} \mathrm{d}x$ is bounded uniformly in n.

By (20) for some $\alpha \in (0{,}1)$. we conclude via the Arzelà–Ascoli theorem that, along a subsequence,

$$m_n \to m, \quad n \to +\infty.$$

Notably, $0 \leq m_n \leq C,$ holds for all n , which yields

$$\int |w_n|^{\gamma'} dx = \int m_n{}^{\gamma'} \frac{|w_n|^{\gamma'}}{m_n{}^{\gamma'-1}} \, dx \leq C^{\gamma'-1} \int m_n \left|\frac{w}{m}\right|^{\gamma'} dx$$

It follows that w_n is bounded in $L^{\gamma'}(T),$ so there exists a subsequence such that

$$w_n \text{ converges to } w.$$

(reflexive property of L_p spaces ensures that every bounded sequence in the space has a weakly convergent subsequence.)

Using (11), we have $||m_n||_{H^{2s-1}_{\gamma'}(T)} \leq C\,||w_n||\Big|_{L^{\gamma'}(T)} \leq C$ so, m_n is bounded in $H^{2s-1}_{\gamma'}$.

So $(m,w) \in \mathscr{E}$ and $liminf \mathscr{E}(m_n, w_n) \geq \mathscr{F}(m,w).$

For the purpose of obtaining a solution for the mean field game (1), we associate the energy constraint from (6) to a dual formulation by employing well-established arguments from convex analysis. For a minimizer (m_k, w_k) of $\mathscr{F}$, we define the convex function on $\mathscr{E}$

$$J\Big(m, w\Big) = \int_T \left[\frac{m}{\gamma'}\left|\frac{w}{m}\right|^{\gamma'} + V(x)m + m_k^q m\right] dx \tag{30}$$

Proposition 4.2 If $(m_{k,} w_k)$ minimizes $\mathscr{F}$ over $\mathscr{E}$, Then

$$\min_{(m,w)\in\mathscr{E}} J(m,w) = J(m_k, w_k). \tag{31}$$

Now we prove Theorem 1.1.

Theorem 4.1. Suppose (m_k, w_k) is a minimizer of J as stated in Lemma 4.1. Then m_k belongs to $H_p^{2s-1}(T),$ for every $p > 1,$ then exists a constant $\lambda \in \mathbb{R}$ and a function $u \in C^{2s+\alpha}(T),$ satisfies (u, λ, m_k) constitutes the solution to (1). And $w_k = -m_k \nabla H(\nabla u).$

Proof. As established in Proposition 4.1, $m_k \in C^{\alpha}(T)$, holds for all $\alpha \in \Big(0{,}2s - 1 - \frac{N}{\gamma'}\Big),$ We introduce the auxiliary functional

$$\mathscr{D}(m, w, u, c) \coloneqq \int_T \left[\frac{m}{\gamma'}\left|\frac{w}{m}\right|^{\gamma'} + V(x)m + m_k^q m - m(-\Delta)^s u + \nabla u \cdot w - cm\right] dx + c.$$

From the identity $\int_T m(-\Delta)^s \varphi dx = \int_T w \cdot \nabla\varphi dx$, we derive

$$J(m_k, w_k) = inf\ \ sup\ \mathscr{D}, \tag{32}$$

This shows that the infimum is attained, i.e., it is a minimum.

Observe that $\mathscr{D}(\cdot,\cdot,u,c)$ is convex, and $\mathscr{D}(m,w,\cdot,\cdot)$ is linear. Owing to the weak lower semicontinuity of $\mathscr{D}(\cdot,\cdot,u,c)$, let us invoke min-max Theorem [7, Theorem 2.3.7], and swap the order of the minimum and the supremum leading to

$$\min sup\,\mathscr{D}\,(m,w,u,c) = \ sup \min \mathscr{D}\,(m,w,u,c). \tag{33}$$

Then between infimum and integral, using Rockafellar's Interchange Theorem [8, 3A. Theorem], and the Legendre transform relationship between H and L, we have

$$\min D\Big(m,w,u,c\Big) = \int \min m\Big[-(-\varDelta)^s u - \tfrac{1}{\gamma}|\nabla u|^{\gamma} + m^{q-1} + V\Big(x\Big) - c\Big]dx + c.\Big]$$

Note that

$$minm[-(-\Delta)^s u - \tfrac{1}{\gamma}|\nabla u|^{\gamma} + m^{q-1} + V(x) - c] =$$
$$\begin{cases} 0, & -(-\Delta)^s u - \tfrac{1}{\gamma}|\nabla u|^{\gamma} + m^{q-1} + V(x) - c \geq 0 \\ -\infty, & -(-\Delta)^s u - \tfrac{1}{\gamma}|\nabla u|^{\gamma} + m^{q-1} + V(x) - c < 0 \end{cases}$$

So, from (33), (34) and (35), we obtain that

$$J\,(m_k, w_k) = sup \int \min m\Big[-(-\varDelta)^s u - \tfrac{1}{\gamma}|\nabla u|^{\gamma} + m^{q-1} + V\Big(x\Big) - c\Big]dx + c.\Big]$$

$$= \sup\{c \in R\ s.t.\ exists\ u \in \ W^{1,\infty}\,(T)\ s.t.\ (-\varDelta)^s u + \tfrac{1}{\gamma}|\nabla u|^{\gamma} + \lambda \leq m^{q-1} + V\Big(x\Big) \tag{34}$$

By Theorem Theorem 3.1, this supremum is indeed a maximum, so there exists a constant $\lambda \in \mathbb{R}$, $u \in C^{2s+\alpha}\,(T) \cap H_p^{2s}\,(T)$, with $\alpha < 2s - 1 - \frac{N}{\gamma'}$ and every $p > 1$, solving

$$(-\varDelta)^s u + \tfrac{1}{\gamma}|\nabla u|^{\gamma} + \lambda = m^{q-1} + V\Big(x\Big) \tag{35}$$

So, equality (33) reads

$$\lambda \ = J\,(m_k, w_k) = \int m_k \left[\tfrac{1}{\gamma'}\left|\tfrac{w}{m}\right|^{\gamma'} + m_k^{q-1} + V\,(x)\right]dx.$$

And by (36) and (10) we get

$$0 = \int m_k \left[\tfrac{1}{\gamma'}\left|\tfrac{w}{m}\right|^{\gamma'} + m_k^{q-1} + V\,(x) - \lambda\right]dx$$

$$= \int m_k \left[\frac{1}{\gamma'}\left|\frac{w}{m}\right|^{\gamma'} + (-\varDelta)^s u + \frac{1}{\gamma}|\nabla u|^{\gamma}\right]dx$$

$$= \int m_k \left[\frac{1}{\gamma'}\left|\frac{w}{m}\right|^{\gamma'} + \frac{1}{\gamma}|\nabla u|^{\gamma} + \nabla u\,\frac{w_k}{m_k}\right]dx$$

Since H is the Legendre transform of L, it follows that

$$\frac{w_k}{m_k} = -|\nabla u|^{\gamma-1}.$$

with $m_k > 0$. Referring back to (10), it follows that m_k serves as a solution to

$$(-\Delta)^s m - \operatorname{div}\left(m|\nabla u|^{\gamma-1}\right) = 0, \quad \text{with} \quad \int_T m dx = 1.$$

Given $mk|\triangledown u|\gamma \in L\infty(T)$, we apply Lemma 2.2 and Lemma 2.3 to deduce $mk \in Hp2s-1(T)$for any $p > 1$. Consequently, (u, λ, mk) constitutes a solution to (1).

5. Conclusion

This paper delves into the existence, uniqueness, and regularity properties of solutions in stationary fractional-order mean-field game systems. By introducing the fractional-order Laplace operator $(-\Delta)^s$, we successfully extend the classical mean-field game framework to stochastic dynamical systems with nonlocal diffusion characteristics. Our research shows that the nonlocality of the fractional-order operator not only alters the coupling mechanism between the Hamilton-Jacobi-Bellman equation and the Fokker-Planck equation but also significantly impacts the tail behavior of the agent distribution density.

The main findings are as follows: Existence and uniqueness of solutions: Under the assumption of appropriate growth conditions and monotonicity of the coupling terms, we prove the existence of stationary solutions using variational methods or fixed-point theory. Nonlocal effects analysis: We explore in detail the moderating effect of the diffusion order $s \in (1/2,1)$ on the system's equilibrium state, revealing the importance of long-range interactions in group decision-making. Regularity estimation: Through refined analytical estimation, we establish the regularity of the system in the Bessel potential space or Holder space, providing theoretical support for subsequent numerical simulations.

This study not only enriches the theoretical framework of nonlocal partial differential equations but also provides new mathematical tools for simulating economic and social science phenomena with anomalous diffusion characteristics. Future research can further explore the long-term behavior of nonstationary (evolutionary) fractional-order MFG systems and the boundary effects on non-compact spaces or complex manifolds.

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