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Harnack Inequality for Solutions of the Inhomogeneous G-Laplace Equation

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Abstract. This paper investigates the local regularity of weak solutions to a class of inhomogeneous generalized Laplace equations where the right-hand side depends on the weak solutions themselves. The core innovation lies in handling the strong coupling between the inhomogeneous term of the equation and the weak solution. We first establish the local boundedness of the solution as a key a priori estimate for the analysis. On this basis, by constructing ingenious test functions and applying delicate iterative techniques, we overcome the additional nonlinear interference caused by the inhomogeneous term, and finally prove the Harnack inequality satisfied by the weak solution in the interior domain. The results of this study provide a theoretical tool for analyzing the more refined properties of related nonlinear problems.

Keywords: Fractional Sobolev Spaces, Harnack Inequality, G-Laplace Operator

1. Introduction

This paper aims to establish a local regularity theory for weak solutions of nonlocal integro-differential equations with non-standard growth properties. We consider the following fractional nonlocal equation defined on a bounded domain $\Omega \subset \mathbb{R}^n \ (n \geqslant 2)$:

$$Lv(x) = f(x, v), \quad x \in \Omega, \tag{1}$$

where the operator $\mathscr{L}$ is defined by

$$Lv(x) \coloneqq \text{P.V.} \int_{\mathbb{R}^n} g\left(\frac{|v(x)-v(y)|}{|x-y|^s}\right) \operatorname{sign}(v(x) - v(y)) \frac{K(x,y)}{|x-y|^s} \, dy. \tag{2}$$

Here $s \in (0,1)$, P.V. denotes "in the sense of principal value", and $g : [0, \infty) \to [0, \infty)$ is a continuous and strictly increasing function satisfying $g(0) = 0$ and $\lim_{t \to \infty} g(t) = \infty$. For $1 < p \leqslant q$, the following inequality holds:

$$1 < p \leqslant \frac{tg(t)}{G(t)} \leqslant q < \infty \quad \text{and} \quad G(t) \coloneqq \int_0^t g(\tau) d\tau. \tag{3}$$

It is worth noting that the kernel function $K : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ is symmetric, i.e., $K(x,y) = K(y,x)$, and satisfies

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$$\frac{\lambda}{|x-y|^n} \leqslant K(x,y) \leqslant \frac{\Lambda}{|x-y|^n}, \qquad x,y \in \mathbb{R}^n, \tag{4}$$

where $0<\lambda \leqslant \Lambda$ are constants. The function f defined on the appropriate Orlicz-Sobolev space satisfies the following assumption:

For all $x \in \mathbb{R}^n$ and $v \in \mathbb{R}$,

$$|f(x,v)| \leqslant c_0|v|^{\beta-1} + h(x), \tag{5}$$

where $1<\beta<np/(n-sp)$, $c_0 \geqslant 0$, and the non-negative function $h\left(x\right) \in L^{G^*}\left(\mathbb{R}^n\right)$.

It is worth noting that the function G is an N-function satisfying the Δ_2 and ∇_2 conditions (see Section 2). Some representative examples of the function $g(\cdot)$ satisfying condition (3) can be found in Reference [1]. When $g(\cdot)$ has a more general structure and the right-hand side of the equation is zero, Equation (1) can be regarded as the nonlocal analog of the classical G-Laplace equation, which has been intensively studied in the calculus of variations and regularity theory. For instance, Fernández Bonder et al. [2] proved the Hölder continuity of weak solutions to the fractional G-Laplace Dirichlet problem, and Chaker, Kim and Weidner [3] studied the interior regularity of nonlocal functionals with (p,q) -growth and the corresponding equations by the De Giorgi class method.

Although abundant achievements have been made in the research of nonlocal problems with non-standard growth, results on their Harnack inequality are still quite limited. For homogeneous equations, recent work such as [4] proved the Harnack inequality under additional structural conditions. Essentially, these conditions require G to have a certain multiplicative or double power growth structure, which restricts the case of general N-functions.

The innovation of this paper is as follows: for the first time, we establish the Harnack inequality for the nonlocal G-Laplace equation with a non-vanishing right-hand side $f(x,u)$ under only the basic structural condition (3) on G. The growth condition satisfied by the right-hand side of the equation brings certain difficulties to the regularity analysis: the integral term $\int |u|^{\beta-1}\varphi dx$ generated by it destroys the scale homogeneity and interferes with the iterative convergence. To solve this difficulty, we develop a method by introducing the generalized Young's inequality and interpolation techniques to transform the term $\int |u|^{\beta-1}$ into an expression related to the mean value of $G(u/r^s)$ with a uniform scale factor r^s, thus making it compatible with the principal term of the equation in the iteration. The results of this paper extend the Harnack inequality for homogeneous equations with power-type growth in [5], and provide a theoretical foundation for studying the regularity of a wider range of nonlocal reaction-diffusion equations.

Combined with the definition of weak solutions, related function spaces and tail properties provided in the next section, we can now state the main research results of this paper.

Theorem 1.1 Let $s \in (0,1)$, $sp \leqslant n$ and $1<\beta \leqslant 2$. Under Assumptions (3), (4) and (5), let $u \in \mathbb{W}^{s,G}\left(\Omega\right) \cap L^g_s\left(\mathbb{R}^n\right)$ be a weak solution of Equation (1.1) satisfying $u \geqslant 0$ in the ball $B_R \equiv B_R(x_0) \Subset \Omega$. Then for any ball $B_{R/2}(x_0) \supset B_r(x_0) \equiv B_r$, the following inequality holds:

$$\sup_{B_r} u \leqslant C \inf_{B_r} u + C r^s g^{-1}\left[r^s\left(\mathrm{Tail}(u_-;x_0,R) + |h|_{L^{G^*}(B_R)}\right)\right],$$

where the constant $C \equiv C(n,s,p,q,\lambda,\Lambda)$.

The structure of this paper is organized as follows: in Section 2, we give the definition of weak solutions to Equation (1) and list some notations and auxiliary inequalities to be used later; Sections 3 and 4 present various

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estimates for weak subsolutions and weak supersolutions of Equation (1), respectively; finally, in Section 5, we prove the Harnack inequality.

2. Preliminaries

This section elaborates on the general assumptions used in the research, gives several basic inequalities and lemmas, explains the function spaces and the concept of weak solutions, and establishes the notations and definitions adopted throughout the paper.

In this paper, we use $c \equiv c(n, p, q, s, \lambda)$ to denote a universal constant determined by the given parameters n, s, p, q, λ, etc., and the value of this constant may vary in different lines. Let $B_r(x_0)$ denote the open ball in $\mathbb{R}^n$ with center $x_0 \in \mathbb{R}^n$ and radius $r > 0$. The ball is abbreviated as B_r when the center is clear from the context. The Lebesgue measure of a measurable set $S \subset \mathbb{R}^n$ is denoted by $|S|$. If a function f is integrable on a set S with $0 < |S| < \infty$, we set

$$(f)_S \coloneqq \fint_S f\,dx \coloneqq \frac{1}{|S|}\int_S f dx.$$

We also define

$$a \vee b \coloneqq \max\{a, b\}, \;\; a_+ \coloneqq \max\{a, 0\}, \;\; a_- \coloneqq -\min\{a, 0\}.$$

A function $G : [0, \infty] \to [0, \infty]$ is called an N-function if it is increasing and convex, and satisfies

$$\lim_{t\to 0^+}\frac{G(t)}{t} = 0, \qquad \lim_{t\to\infty}\frac{G(t)}{t} = \infty, \;\; G\left(0\right) = 0, \qquad G\left(1\right) = 1.$$

The conjugate function of G is defined by

$$G^*(t) \coloneqq \sup_{s\geqslant 0}(st - G(s)), \quad t \geqslant 0.$$

We always assume that $G \in C^1\left([0, \infty)\right)$ satisfies Equation (3). From (3) and Reference [6], we give the following known inequalities:

(1) For $t \in [0, \infty)$,

$$a^q G(t) \leqslant G(at) \leqslant a^p G(t) \;\; if\ a \in (0,1) \;\; and \;\; a^p G(t) \leqslant G(at) \leqslant a^q G(t) \;\; if\ a \in (1, \infty), \tag{6}$$

and

$$a^{p'} G^*(t) \leqslant G^*(at) \leqslant a^{q'} G^*(t) \text{ if } a \in (0,1) \text{ and } a^{q'} G^*(t) \leqslant G^*(at) \leqslant a^{p'} G^*(t) \text{ if } a \in (1, \infty), \tag{7}$$

where p', q' are the corresponding Hölder conjugate exponents ($\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$).

(2) For any $\varepsilon \in (0,1)$, we have the following Young's inequality:

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$$ts \leqslant \epsilon^{1-q} G(t) + \epsilon G^*(s), \quad t, s \geqslant 0. \tag{8}$$

(3) For $t \geqslant 0$,

$$G^*(g(t)) = tg(t) - G(t) \leqslant (q-1)G(t). \tag{9}$$

Its convexity implies that

$$2^{-1}(G(t) + G(\tau)) \leqslant G(t+\tau) \leqslant 2^{q-1}(G(t) + G(\tau)). \tag{10}$$

In addition, the N-function G satisfies the Δ_2 condition and the ∇_2 condition (see [7, Proposition 2.3]):
(Δ_2) There exists $\mu > 1$ such that $G(2t) \leqslant \mu G(t)$ for all $t \geqslant 0$;
(∇_2) There exists $\nu > 1$ such that $G(t) \leqslant \frac{1}{2\nu} G(\nu t)$ for all $t \geqslant 0$,
where μ and ν only depend on p, q . Note that the ∇_2 condition is exactly the Δ_2 condition imposed on G^* .

Next, we introduce the concept of the fractional Orlicz-Sobolev space. For a given domain $\Omega \in \mathbb{R}^n$ and an N-function G satisfying the Δ_2 and ∇_2 conditions, the Orlicz space $L^G(\Omega)$ is defined by

$$L^G(\Omega) \coloneqq \left\{ v : \Omega \to \mathbb{R} \text{ is a measurable function: } \int_\Omega G(|u(x)|)dx < \infty \right\},$$

which is also a Banach space under the Luxemburg norm.

$$\left| v \right|_{L^G(\Omega)} \coloneqq \inf \left\{ \lambda > 0 : \int_\Omega G\left(\frac{|v(x)|}{\lambda} \right) dx \leqslant 1 \right\}.$$

Then

$$|v|_{L^G(\Omega)} \leqslant \int_\Omega G(|v|)dx + 1. \tag{11}$$

For $s \in (0,1)$, the fractional Orlicz-Sobolev space $W^{s,G}(\Omega)$ is defined by

$$W^{s,G}(\Omega) \coloneqq \left\{ v \in L^G(\Omega) : \int_\Omega \int_\Omega G\left(\frac{|v(x) - v(y)|}{|x-y|^s} \right) \frac{dxdy}{|x-y|^n} < \infty \right\},$$

endowed with the norm

$$|v|_{W^{s,G}(\Omega)} \coloneqq |v|_{L^G(\Omega)} + [v]_{s,G;\Omega},$$

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where $[v]_{s,G;\Omega}$ denotes the Gagliardo semi-norm,

$$[v]_{s,G;\Omega} \coloneqq \inf\left\{\lambda > 0 : \int_{\Omega}\int_{\Omega} G\left(\frac{|v(x) - v(y)|}{\lambda|x - y|^{s}}\right)\frac{dxdy}{|x - y|^{n}} \leqslant 1\right\}.$$

We now set $C_\Omega \coloneqq (\Omega \times \mathbb{R}^n) \cup (\mathbb{R}^n \times \Omega)$. For a measurable function v on $\mathbb{R}^n$, we define another function space related to the weak solutions of Equation (1) as follows:

$$\mathbb{W}^{s,G}(\Omega) \coloneqq \left\{v : \mathbb{R}^n \to \mathbb{R}\ |v|_\Omega \in L^G(\Omega), \iint_{C_\Omega} G\left(\frac{|v(x) - v(y)|}{|x - y|^{s}}\right)\frac{dxdy}{|x - y|^{n}} < \infty\right\},$$

Note that if $v \in W^{s,G}(\Omega)$, then $v|_\Omega \in W^{s,G}(\Omega)$.

Next, we introduce a special quantity that plays a core role in dealing with nonlocal operators:

$$L_s^g(\mathbb{R}^n) \coloneqq \left\{\upsilon : \mathbb{R}^n \to \mathbb{R}\ \int_{\mathbb{R}^n} g\left(\frac{|\upsilon(x)|}{(1 + |x|)^{s}}\right)\frac{dx}{(1 + |x|)^{n+s}} < \infty\right\}.$$

For $v \in L_s^g(\mathbb{R}^n)$, its nonlocal tail on the ball $B_R(x_0)$ is defined by

$$\mathrm{Tail}(\upsilon; x_0, R) \coloneqq \int_{\mathbb{R}^n \setminus B_R(x_0)} g\left(\frac{|\upsilon(x)|}{|x - x_0|^{s}}\right)\frac{dx}{|x - x_0|^{n+s}}.$$

Note that $v \in L_s^g(\mathbb{R}^n)$ if and only if $\mathrm{Tail}(v; x_0, R) < \infty$ for all $x_0 \in \mathbb{R}^n$ and $R > 0$. To avoid ambiguity, g^{-1} and G^{-1} denote the inverse functions of g and G , respectively.

Next, we recall the function g satisfying (3) and the kernel function K satisfying (1.4) to define the weak solution of Equation (1).

Definition 2.1 Assume that f satisfies condition (5) and let $s \in (0,1)$. A function $v \in \mathbb{W}^{s,G}(\Omega) \cap L_s^g(\mathbb{R}^n)$ is called a weak subsolution (weak supersolution, correspondingly) of (1) if for all non-negative functions $\eta \in \mathbb{W}^{s,G}(\Omega)$ (satisfying $\eta = 0$ in $\mathbb{R}^n \setminus \Omega$), the following inequality holds:

$$\iint_{C_\Omega} g\left(\frac{|v(x) - v(y)|}{|x - y|^{s}}\right)\mathrm{sign}(v(x) - v(y))(\eta(x) - \eta(y))K(x, y)\frac{dxdy}{|x - y|^{s}} \leqslant (\geqslant, \text{resp.})$$
$$\int_\Omega f(x, v)\eta(x)dx,$$

A function v is called a weak solution of (1) if it is both a weak subsolution and a weak supersolution, i.e., for all $\eta \in \mathbb{W}^{s,G}(\Omega)$ (satisfying $\eta = 0$ in $\mathbb{R}^n \setminus \Omega$),

$$\iint_{C_\Omega} g\left(\frac{|v(x) - v(y)|}{|x - y|^{s}}\right)\mathrm{sign}(v(x) - v(y))(\eta(x) - \eta(y))K(x, y)\frac{dxdy}{|x - y|^{s}} = \int_\Omega f(x, v)\eta(x)dx,$$

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At the end of this section, we list some classical tools that will be used in proving the main results of this paper.

We first give the integral form of the following fractional Sobolev-Poincaré inequality.

Lemma 2.1 ([5, Lemma 2.2]) Let $s \in (0,1)$. Then there exists a constant $\theta = \theta(n, s) > 1$ such that if G is an N-function satisfying the Δ_2 condition and the ∇_2 condition (with constants μ and ν respectively), and $f \in \mathbb{W}^{s,G}(B_r)$, then

$$⨍_{B_r}\left(G\left(\frac{|f-(f)_{B_r}|}{r^s}\right)^{\theta}dx\right)^{\frac{1}{\theta}} \leqslant c ⨍_{B_r} ⨍_{B_r} G\left(\frac{|f(x)-f(y)|}{|x-y|^s}\right)\frac{dydx}{|x-y|^n}, \tag{12}$$

where $c = c(n, s, \mu, \nu) > 0$.

Next, we recall the following Poincaré-type inequality.

Lemma 2.2 ([8, Corollary 6.2]) Let $U \subset \mathbb{R}^n$ be a bounded open set with a Lipschitz boundary. Assume that G is an N-function satisfying (3). Then there exists a constant $c > 0$, depending on n, s, p, q and U , such that for each $s \in (0,1)$ and $u \in W^{s,G}(U)$,

$$\int_U G(|u|)dx \leqslant c\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} G\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^n}.$$

The following is the iterative lemma.

Lemma 2.3 ([9, Lemma 7.1]) Let $\beta > 0$, and let A_i be a sequence of positive real numbers satisfying

$$A_{i+1} \leqslant CB^i A_i^{1+\beta},$$

where $C > 0$, $B > 1$. If $A_0 \leqslant C^{-\frac{1}{\beta}} B^{-\frac{1}{\beta^2}}$, then

$$A_i \leqslant B^{-\frac{i}{\beta}} A_0,$$

in particular,

$$\lim_{i\to\infty} A_i = 0.$$

3. Auxiliary estimates

In this section, we recall several estimates for weak solutions of Equation (1), and these results can be found in [10, Section 3]. First is the Caccioppoli estimate with tail terms, see [10, Proposition 3.1].

Lemma 3.1 Let f satisfy (5). Under Conditions (3) and (4), take $r > 0$, $B_r \equiv B_r(x_0) \Subset \Omega$, and let $u \in \mathbb{W}^{s,G}(\Omega) \cap L_s^g(\mathbb{R}^n)$ be a weak solution of Equation (1). Then for any $k \geqslant 0$ and non-negative function $\phi \in C_0^\infty(B_r)$ satisfying $0 \leqslant \phi \leqslant 1$,

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$$
\begin{aligned}
&\int_{B_r}\int_{B_r} G\left(\tfrac{|w_\pm(x)-w_\pm(y)|}{|x-y|^s}\right)\min\left\{\Phi^q(x),\Phi^q(y)\right\}\tfrac{dxdy}{|x-y|^n}\\
&\leqslant c\|h\|_{L^{G^*}(B_r)}\int_{B_r}\int_{B_r} G\left(\tfrac{|\Phi(x)-\Phi(y)|}{|x-y|^s}\max\left\{w_\pm(x),w_\pm(y)\right\}\right)\tfrac{dx\,dy}{|x-y|^n}\\
&+c\int_{B_r} w_\pm(x)\Phi^q(x)dx\sup_{y\in\operatorname{supp}\Phi}\left(\int_{\mathbb{R}^n\setminus B_r} g\left(\tfrac{w_\pm(x)}{|x-y|^s}\right)\tfrac{dx}{|x-y|^{n+s}}\right)\\
&+c\int_{B_r}\left(w_\pm^\beta(x)+|u(x)|^\beta\right)\max\left\{\chi_{\{u\geq\ \}}(x),\chi_{\{\ \geq u\}}(x)\right\}\Phi^q(x)dx,
\end{aligned}
\tag{13}
$$

where $w_\pm := (u-\ )_\pm$, and the constant $c > 0$ depends on $n,s,p,q,c_0,\beta,\lambda$ and Λ .

The following lemma is a modification of [10, Theorem 3.1].

Lemma 3.2 (Local Boundedness) Let $u\in\mathbb{W}^{s,G}(\Omega)\cap L_s^g(\mathbb{R}^n)$ be a weak subsolution of (1.1), and $B_r(x_0)\Subset\Omega$ with $r\in$ (0,1). Assume that Conditions (3), (4) and (5) hold. For $1<\beta\leq p$,

$$
\sup_{B_{r/2}} u_+ \leq \bar{c}\, r^s G^{-1}\left(\delta^{\frac{\theta}{1-\theta}} \fint_{B_r} G\left(\tfrac{u_+}{r^s}\right)dx\right)\vee 1 + r^s g^{-1}\left(\delta r^s \operatorname{Tail}\left(u_+;x_0,\tfrac{r}{2}\right)\right)
\tag{14}
$$

where the constant $\bar{c}\equiv\bar{c}\left(n,s,p,q,\lambda,\Lambda,\beta,c_0,\|h\|_{L^{G^*}(B_r)},\mu\right)>0$.

Proof Let $k > 0$ be a constant to be determined in the following proof. For each $j\in\mathbb{N}_0$, define

$$
r_j=\left(1+2^{-j}\right)\frac{r}{2},\ B_j=B_{r_j}(x_0),\ k_j=\left(1-2^{-j}\right)k,\ w_j=\left(u-k_j\right)_+
$$

$$
\text{and}\quad A_j \coloneqq G^{-1}\left(\frac{k}{r^s}\right)\fint_{B_r} G\left(\frac{w_j}{r^s}\right)dx.
$$

Following the proof idea of [10, Theorem 3.1] and using Lemma 3.1, we obtain

$$
A_{j+1}\leqslant c_1 2^{(n+sq+2q+\beta)j\theta}\left[1+r^{s\beta}+\frac{r^s}{g\left(\frac{k}{r^s}\right)}\operatorname{Tail}\left(u_+;x_0,\frac{r}{2}\right)\right]^{\theta} A_j^{\theta},
$$

where $c_1\equiv c_1\left(n,s,p,q,\lambda,\Lambda,\mu,\|h\|_{L^{G^*}(B_r)},\mu\right)\geqslant 1$, and $\theta\equiv\theta(n,s)>1$ is the constant in Lemma 2.1.

Now set

$$
k=r^s g^{-1}\left(\delta r^s\operatorname{Tail}\left(u_+;x_0,\frac{r}{2}\right)\right)+r^s G^{-1}\left(c_2\delta^{\frac{\theta}{1-\theta}}\fint_{B_r} G\left(\frac{u_+}{r^s}\right)dx\right)\vee 1,
$$

where $c_2=\left[c_1\left(2+r^{s\beta}\right)^{\theta}\right]^{\frac{1}{\theta-1}} 2^{\frac{(n+sq+2q+\beta)\theta}{(\theta-1)^2}}$. Let $\bar{c}=2+r^{s\beta}$, then

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$$A_{j+1} \leqslant c_1 \overline{c}^{\theta} 2^{(n+sq+2q+\beta)\theta j} A_j^{\theta} \quad \text{和} \quad A_0 \leqslant c_2^{-1} = \left[c_1 \overline{c}^{\theta}\right]^{\frac{-1}{\theta-1}} 2^{\frac{-(n+sq+2q+\beta)\theta}{(\theta-1)^2}}.$$

Therefore, by Lemma 2.3, we have $\lim_{j\to\infty} A_j = 0$, which implies $\sup_{B_{r/2}} u_+ \leqslant k$, i.e., Equation (14) holds.

4. Propagation of positivity

This section aims to prove a series of lemmas to lay the foundation for the proof of the Harnack inequality.

Proposition 4.1 Under Assumptions (3), (4) and (5), let $u \in \mathbb{W}^{s,G}(\Omega) \cap L_s^g(\mathbb{R}^n)$ be a weak supersolution of Equation (1) satisfying $u \geqslant 0$ in $B_R \equiv B_R(x_0) \Subset \Omega$. Let $k \geqslant 0$, $sp \leqslant n$ and $1 < \beta \leqslant p$. Assume that there exist $\sigma \in (0,1)$ and some $0 < r < R/16$ such that

$$|B_r \cap \{u \geqslant k\}| \geqslant \sigma |B_r| \tag{15}$$

Then there exists a constant $\delta \equiv \delta(n, s, p, q, \lambda, \Lambda, \sigma) \in (0,1/4)$ satisfying

$$\inf_{B_{4r}} u \geqslant \delta k - r^s g^{-1}(r^s Tail(u_-; x_0, R)). \tag{16}$$

Before proving this proposition, we first need the following lemma.

Lemma 4.1 Under Assumptions (3), (4) and (5), let $k \geqslant 0$, $sp \leqslant n$, $1 < \beta \leqslant p$ and let $u \in \mathbb{W}^{s,G}(\Omega) \cap L_s^g(\mathbb{R}^n)$ be a weak supersolution of Equation (1) satisfying $u \geqslant 0$ in $B_R(x_0) \Subset \Omega$. If there exists $\sigma \in (0,1)$ such that

$$|B_r \cap \{u \geqslant k\}| \geqslant \sigma |B_r|, \tag{17}$$

where $0 < r < R/16 \leqslant 1$, then it can be deduced that: for any $\delta \in \left(0, \frac{1}{2}\right)$,

$$\left|B_{6r} \cap \left\{u \leqslant 2\delta k - r^s g^{-1}(r^s \mathrm{T}ail(u_-; x_0, R))\right\}\right| \leqslant \frac{C}{\sigma \log \frac{1}{2\delta}} |B_{6r}|,$$

where $C > 0$ only depends on n, p, q, s, Λ.

Proof Let $v(x) := u(x) + d$, where $d = r^s g^{-1}(r^s Tail(u_-; x_0, R))$. Take a cut-off function $\eta \in C_0^\infty(B_{7r})$ such that

$$0 \leqslant \varphi \leqslant 1, \quad |\nabla\varphi| \leqslant \frac{c}{r}, \text{and } \varphi \equiv 1 \text{ in } B_{6r}.$$

We choose $\eta := \frac{v\varphi^q}{G(v/r^s)}$ as the test function, and obtain

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$$\begin{aligned}
&\int_{\mathbb{R}^n} f\left(x,v\right)\tfrac{v(x)\varphi^q(x)}{G(v(x)/r^s)}\,dx \\
&\qquad \leqslant \int_{B_{8r}}\int_{B_{8r}} g\left(\tfrac{|v(x)-v(y)|}{|x-y|^s}\right)\mathrm{sign}\left(v\left(x\right)-v\left(y\right)\right)\left(\tfrac{v(x)\varphi^q(x)}{G(v(x)/r^s)}-\tfrac{v(y)\varphi^q(y)}{G(v(y)/r^s)}\right)\tfrac{K(x,y)}{|x-y|^s}\,dxdy \\
&\qquad +2\int_{\mathbb{R}^n\setminus B_{8r}}\int_{B_{8r}} g\left(\tfrac{|v(x)-v(y)|}{|x-y|^s}\right)\mathrm{sign}\left(v\left(x\right)-v\left(y\right)\right)\tfrac{v(x)\varphi^q(x)}{G(v(x)/r^s)}\,\tfrac{K(x,y)}{|x-y|^s}\,dxdy \\
&=:\ I+\ II.
\end{aligned}$$

Following the argument in [4, Lemma 3.2], we have

$$I\leqslant -\tfrac{1}{C}\int_{B_{8r}}\int_{B_{8r}}\left|\log v\left(x\right)-\log v\left(y\right)\right|\tfrac{dxdy}{|x-y|^n}+Cr^n. \tag{19}$$

$$II\leqslant Cr^n. \tag{20}$$

Using the definition of the function ϕ and the condition satisfied by f, the left-hand side of Equation (18) is estimated as follows:

$$\begin{aligned}
&\int_{\mathbb{R}^n} f\left(x,v\right)\tfrac{v(x)\varphi^q(x)}{G(v(x)/r^s)}\,dx \\
&\quad =\int_{B_{8r}} f(x,v)_+\tfrac{v(x)\phi^q(x)}{G(v(x)/r^s)}\,dx-\int_{B_{8r}} f(x,v)_-\tfrac{v(x)\phi^q(x)}{G(v(x)/r^s)}\,dx \\
&\quad \geqslant \int_{B_{8r}} f(x,v)_+\tfrac{v(x)}{G(v(x)/r^s)}\,dx-\|h\|_{L^{G^*}(B_{8r})}\int_{B_{8r}} G\left(\tfrac{v(x)}{G(v(x)/r^s)}\right)dx-\int_{B_{8r}}\tfrac{v^\beta(x)}{G(v(x)/r^s)}\,dx.
\end{aligned} \tag{21}$$

Since $\frac{v(x)}{G(v(x)/r^s)}\in\mathbb{W}^{s,G}\left(\Omega\right)$, combining (19), (20) and (21) yields:

$$\int_{B_{8r}}\int_{B_{8r}}\left|\log v\left(x\right)-\log v\left(y\right)\right|\tfrac{dxdy}{|x-y|^n}\leqslant cr^n+\int_{B_{8r}}\tfrac{v^\beta(x)}{G(v(x)/r^s)}\,dx, \tag{22}$$

where the constant c depends on $n,\ p,\ q,\ s,\ \Lambda,\lambda$ and $\|h\|_{L^{G^*}(B_{8r})}$. We now estimate the second term on the right-hand side of the above equation. Given $t>1$, combining the Δ_2 condition of the N-function G with Equation (10), there exists $\mu>1$ such that the following inequality holds:

$$t^pG\left(2\right)\leqslant G\left(2t\right)\leqslant \mu G\left(t\right)\leqslant \mu^2G\left(\frac{1}{2}t\right)\leqslant \mu^2t^qG\left(\frac{1}{2}\right).$$

It is equivalent to

$$c_1t^p\leqslant G\left(t\right)\leqslant c_2t^q, \tag{23}$$

where c_1,c_2 depend on $n,\ \mu$. From (23), for $1<\beta\leq p$, we get

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$$v^{\beta} \leqslant cG\,(v/r^{s})r^{s\beta}.$$

Therefore, Equation (22) can be simplified to

$$\int_{B_{8r}}\int_{B_{8r}} |\log v\,(x) - \log v\,(y)|\,\frac{dxdy}{|x-y|^{n}} \leqslant cr^{n}.$$

For any $\delta \in (0{,}1/4)$, let

$$w = \min\left\{(\log\,(k+d) - \log\,(u+d))_{+},\ \log\frac{1}{2\delta}\right\},$$

and we observe that

$$\int_{B_{6r}} \left|w\,(x) - (w)_{B_{6r}}\right|dx \leqslant C\,(n)\int_{B_{6r}}\int_{B_{6r}} |w\,(x) - w\,(y)|dxdy.$$

Thus

$$\int_{B_{6r}} \left|w\,(x) - (w)_{B_{6r}}\right|dx \leqslant c\int_{B_{6r}}\int_{B_{6r}} |\log v\,(x) - \log v\,(y)|\,\frac{dxdy}{|x-y|^{n}} \leqslant cr^{n}. \tag{24}$$

Since $\{w = 0\} = \{u \geqslant k\}$, from Condition (17) we have

$$|B_{6r} \cap \{w = 0\}| \geqslant |B_{r} \cap \{w = 0\}| \geqslant \sigma\,|B_{r}| = \frac{\sigma}{6^{n}}\,|B_{6r}|.$$

Then

$$\log\frac{1}{2\delta} = ⨍_{B_{6r}\cap\{w=0\}}\left(\log\frac{1}{2\delta} - w\right)dx \leqslant \frac{c}{\sigma}⨍_{B_{6r}}\left(\log\frac{1}{2\delta} - w\right)dx = \frac{c}{\sigma}\left(\log\frac{1}{2\delta} - (w)_{B_{6r}}\right).$$

Integrating the above inequality over the region $B_{6r} \cap \left\{w = \log\frac{1}{2\delta}\right\}$ and using (24), we get

$$\left|B_{6r} \cap \left\{w = \log\frac{1}{2\delta}\right\}\right|\log\frac{1}{2\delta} \leqslant \frac{c}{\sigma}\int_{B_{6r}\cap\{w=\log\frac{1}{2\delta}\}} \left|w - (w)_{B_{6r}}\right|dx \leqslant \frac{c}{\sigma}\int_{B_{6r}} \left|w - (w)_{B_{6r}}\right|dx$$

$$\leqslant \frac{\bar{c}}{\sigma}\,|B_{6r}|,$$

where $\overline{c} \equiv \overline{c}\left(n, s, p, q, \lambda, \Lambda, \|h\|_{L^{G^*}(B_{8r})}\right)$. From the definition of w, we have:

$$|B_{6r} \cap \{u \leqslant 2\delta k - d\}| \leqslant |B_{6r} \cap \{u + d \leqslant 2\delta (k + d)\}| \leqslant \frac{\overline{c}}{\sigma \log \frac{1}{2\delta}} |B_{6r}|.$$

Proof of Proposition 4.1 Assume that for all u,

$$\delta k > 2r^s g^{-1} (r^s Tail (u_-; x_0, R)),$$

otherwise the conclusion is trivial. For each $i \in \mathbb{N} \cup \{0\}$, we define

$$\begin{aligned} \rho_i = \left(4 + \tfrac{1}{2^{i-1}}\right) r, \ \ \widetilde{\rho_i} = \tfrac{\rho_i + \rho_{i+1}}{2}, \ \ B_i = B_{\rho_i}, \ \ \widetilde{B_i} = B_{\widetilde{\rho_i}}, \\ l_i = \left(1 + \tfrac{1}{2^{i+1}}\right) \delta k \qquad \text{and} \qquad w_i = (l_i - u)_+. \end{aligned}$$

Take $\phi_i \in C_0^\infty\left(\widetilde{B_i}\right)$ such that $\phi_i \equiv 1$ on $\widetilde{B}_{i+1}$, satisfying $0 \leqslant \phi_i \leqslant 1$ and its gradient satisfies $|D\phi_i| \leqslant 2^{i+3}/r$. Replacing w_-, B_r and ϕ with w_i, $\widetilde{B_i}$ and ϕ_i respectively, from the proof process of Lemma 3.1, we obtain

$$\begin{aligned} &⨍_{B_{i+1}} \int_{B_{i+1}} G\left(\tfrac{|w_i(x) - w_i(y)|}{|x-y|^s}\right) \tfrac{dxdy}{|x-y|^n} \\ &\leq c\|h\|_{L^{G^*}(B_i)} ⨍_{B_i} \int_{B_i} G\left(\tfrac{|\phi_i(x) - \phi_i(y)|}{|x-y|^s} \max\{w_i(x), w_i(y)\}\right) \tfrac{dxdy}{|x-y|^n} \\ &+c ⨍_{B_i} w_i(x) \phi_i^q(x)\, dx \left(\sup_{y \in \widetilde{B_i}} \int_{\mathbb{R}^n \setminus B_i} g\left(\tfrac{w_i(x)}{|x-y|^s}\right) \tfrac{dx}{|x-y|^{n+s}}\right) \\ &+c ⨍_{B_i} |u(x)|^{\beta - 1} w_i(x) \phi^q(x) dx. \end{aligned} \tag{25}$$

According to [5, Lemma 4.1], using (10) and Jensen's inequality and combining with (25), we get:

$$\begin{aligned} I &:= \left(⨍_{B_{i+1}} G^\theta\left(\tfrac{w_i}{r^s}\right) dx\right)^{\frac{1}{\theta}} \\ &\leqslant c ⨍_{B_i} \int_{B_i} G\left(\tfrac{|\phi_i(x) - \phi_i(y)|}{|x-y|^s} \max\{w_i(x), w_i(y)\}\right) \tfrac{dxdy}{|x-y|^n} \\ &+c ⨍_{B_i} w_i(x) \phi_i^q(x)\, dx \left(\sup_{y \in \widetilde{B_i}} \int_{\mathbb{R}^n \setminus B_i} g\left(\tfrac{w_i(x)}{|x-y|^s}\right) \tfrac{dx}{|x-y|^{n+s}}\right) + c ⨍_{B_i} G\left(\tfrac{w_i}{r^s}\right) dx \\ &+c ⨍_{B_i} |u(x)|^{\beta - 1} w_i(x) \phi^q(x) dx. \\ &=: II + III + IV + V. \end{aligned} \tag{26}$$

Define

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$$A_i = \frac{|B_i \cap \{u < l_i\}|}{|B_i|}.$$

Through the definition of $K(x,y)$ and combining with [5, Lemma 4.1], the estimates of I, II, III and IV are as follows:

$$I \geqslant A_{i+1}^{\frac{1}{\theta}} G\left(\frac{l_i - l_{i+1}}{r^s}\right), \;\; II \leqslant c2^{qi} G\left(\frac{l_i}{r^s}\right) A_i,$$

$$III \leqslant c2^{(n+sq)i} G\left(\frac{l_i}{r^s}\right) A_i, \;\; IV \leqslant cG\left(\frac{l_i}{r^s}\right) A_i.$$

We now estimate the term V. If $u > l$, this term is zero. Therefore, let $l > u$, then:

$$⨍_{B_i} |u(x)|^{\beta-1} w_i(x) \phi^q(x) dx \leqslant cl^{\beta} \frac{|B_i \cap \{u < l_i\}|}{|B_i|} \leqslant cr^{s\beta} G\left(\frac{l_i}{r^s}\right) A_i.$$

Substituting the estimates of I, II, III, IV and V into (26) and using

$$\frac{G(l_i/r^s)}{G((l_i - l_{i+1})/r^s)} \leqslant \left(\frac{l_i}{l_i - l_{i+1}}\right)^q \leqslant 2^{q(i+3)},$$

we obtain

$$A_{i+1} \leqslant c_0 2^{(n+sq+3q)\theta i} A_i^{\theta},$$

where $c_0^{\frac{1}{\theta}} = c\left(1 + r^{s\beta}\right)$. Applying Lemma 2.3, if

$$A_0 \leqslant c_0^{-\frac{1}{\theta-1}} 2^{-\frac{(n+sq+3q)\theta}{(\theta-1)^2}} =: \alpha,$$

then $A_i \to 0$ as $i \to \infty$. We now verify this condition.

$$A_0 = \frac{\left|B_{6r} \cap \left\{u < \frac{3}{2}\delta k\right\}\right|}{|B_{6r}|} \leqslant \frac{\left|B_{6r} \cap \left\{u < 2\delta k - r^s g^{-1} Tail(u_-; x_0, R)\right.\right.}{|B_{6r}|} \leqslant \frac{\bar{c}}{\sigma \log \frac{1}{2\delta}}.$$

It suffices to choose an appropriate $\delta \in (0, 1/4)$ such that

$$\frac{C}{\sigma \log \frac{1}{2\delta}} \leqslant \alpha \quad \Longrightarrow \quad \delta \leqslant \frac{1}{2} e^{-\frac{C}{\sigma\alpha}} < \frac{1}{2},$$

then the desired conclusion $\lim_{j\to\infty} A_j = 0$ holds. That is, there exists a constant δ only depending on n, p, q, s, Λ and σ such that

$$u(x) \geqslant \delta k.$$

holds in B_{4r} .

5. Conclusion

In this section, we establish the nonlocal Harnack inequality. We first prove the following lemma.

Lemma 5.1 Under Conditions (3), (4) and (5), let $u \in \mathbb{W}^{s,G}(\Omega) \cap L_s^g(\mathbb{R}^n)$ be a weak solution of Equation (1) satisfying $u \geqslant 0$ in $B_R(x_0) \Subset \Omega$. Let $1 < \beta \leqslant 2$, then for any $0 < r < R$, the following inequality holds:

$$\operatorname{Tail}(u_+; x_0, r) \leq c\left[r^{-s} g\left(r^{-s} \sup_{B_r} u\right) + Tail\left(u_-; x_0, R\right) + \|h\|_{L^{G^*}(B_R)}\right],$$

where the constant $c > 0$ only depends on $\lambda, \Lambda, n, p, q, s$.

Proof

Let $k := \sup_{B_r} u$, and we may assume without loss of generality that $k > 0$. If $k = 0$, then $u \equiv 0$ in B_r , and the Harnack inequality holds trivially in this case. Take the test function

$$\eta \coloneqq (u - 2k)\varphi^q,$$

where $\varphi \in C_0^\infty(B_r)$ satisfies

$$0 \leqslant \varphi \leqslant 1, \quad \varphi \equiv 1 \text{ in } B_{\frac{r}{2}}, \quad \varphi \equiv 0 \text{ in } \mathbb{R}^n \setminus B_{\frac{3r}{4}}, \quad \text{and} \quad |\nabla\varphi| \leqslant \frac{c}{r},$$

We thus obtain

$$\begin{aligned} I &= \int_{B_r} f(x,u)\eta(x)dx \\ &= \int_{B_r}\int_{B_r} g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\operatorname{sign}(u(x)-u(y))(\eta(x)-\eta(y))K(x,y)\frac{dxdy}{|x-y|^s} \\ &\quad +2\int_{\mathbb{R}^n\setminus B_r}\int_{B_r} g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\operatorname{sign}(u(x)-u(y))\eta(x)K(x,y)\frac{dxdy}{|x-y|^s} \\ &=: II + III. \end{aligned} \tag{27}$$

Following the argument in [5, Lemma 5.1], we have

$$II \geqslant -CG\left(k/r^{s}\right)\left|B_{r}\right| \geqslant -ckr^{-s}g\left(k/r^{s}\right)\left|B_{r}\right|. \tag{28}$$

$$III \geqslant Ck\left|B_{r}\right|Tail\left(u_{+};x_{0},r\right) - Ckr^{-s}g\left(k/r^{s}\right)\left|B_{r}\right| - Ck\left|B_{r}\right|Tail\left(u_{-};x_{0},R\right). \tag{29}$$

For the integral term I, note that for all $\forall t \geqslant 0$ and $0 \leqslant m \leqslant 1$, $t^{m} \leqslant 1+t$ holds. If $1 < \beta \leqslant 2$, from (5) and $G(1) = 1$, we get:

$$\begin{aligned} I &= \textstyle\fint_{B_r} f(x,u)(u-2k)\varphi^{q}(x)dx \\ &\leqslant \textstyle\fint_{B_r} u^{\beta-1}(u-2k)\varphi^{q}(x)dx + \fint_{B_r} h(x)(u-2k)\varphi^{q}(x)dx \\ &\leqslant \textstyle\fint_{B_r}(u-2k)\varphi^{q}(x)dx + \fint_{B_r} u(u-2k)\varphi^{q}(x)dx + k\left|B_{r}\right|\|h\|_{L^{G^{*}}(B_R)}G\left(\varphi^{q}(x)\right) \\ &\leqslant k\left|B_{r}\right| + k\left|B_{r}\right|\|h\|_{L^{G^{*}}(B_R)}. \end{aligned} \tag{30}$$

Substitute (28), (29) and (30) into (29), and divide both sides of the equation by $k\left|B_{r}\right|$. The desired estimate is then established.

We now prove the main result of this paper.

Proof of Theorem 1.1

Fix the ball $B_R \equiv B_R(x_0)$, and let $\gamma = \frac{\theta}{1-\theta}$. From Lemma 3.2 and Lemma 5.1, for any ball $B_\rho \equiv B_\rho(x_0) \Subset \Omega$, the following inequality holds:

$$\sup_{B_{\rho/2}} u \leqslant c\delta^{-\gamma}\rho^{s}G^{-1}\left(\fint_{B_\rho} G\left(\frac{u}{\rho^{s}}\right)dx\right) \vee 1 + c\delta \sup_{B_\rho} u$$

$$+c\delta\rho^{s}g^{-1}\left[\rho^{s}\left(\mathrm{Tail}\left(u_{-};x_{0},R\right) + \|h\|_{L^{G^{*}}(B_R)}\right)\right].$$

Following the proof idea of [5, Theorem 1.1], we finally obtain the validity of the following inequality:

$$\begin{aligned} \sup_{B_r} u &\leqslant c\left(\fint_{B_r} u^{t}dx\right)^{\frac{1}{t}} \vee 1 + cr^{s}g^{-1}\left[r^{s}\left(\mathrm{Tail}\left(u_{-};x_{0},R\right) + \|h\|_{L^{G^{*}}(B_R)}\right)\right] \\ &\leqslant c\inf_{B_r} u + cr^{s}g^{-1}\left(r^{s}\mathrm{Tail}\left(u_{-};x_{0},R\right)\right) + cr^{s}g^{-1}\left(r^{s}\|h\|_{L^{G^{*}}(B_R)}\right). \end{aligned}$$

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