Proceedings of the 3rd International Conference on Mathematical Physics and Computational Simulation
DOI: 10.54254/2753-8818/2025.31307

Ramanujan Summation, Zeta Function and Divergent Series

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Abstract. This essay explores the interrelationship between Ramanujan summation, the Riemann zeta function, and divergent series. the connection between Ramanujan summation, the Riemann zeta function, and divergent series will be explored. My objective is to illustrate how Ramanujan summation can be interpreted in a way to assign values to divergent series, thus extending classical convergence and gaining better understanding of infinite sums. Originally going back to the development of convergent series the Riemann zeta function is a fundamental element in analytic number theory and produces an architecture for understanding how prime numbers are distributed. Finally, I test the accuracy of the Ramanujan series and the analytical equation by presenting the special divergent series from the sum of all natural numbers. It may not be an intuitive outcome from a more elementary viewpoint, but it has been obtained in the overseas of some complicated mathematics and appears in the application of theory physics, for example; cable theory and Casimir effect computations involving infinite series sums that naturally emerge in quantum field theory.

Keywords: Divergent Series, Ramanujan summation, Riemann-Euler Zeta Function, Gamma Function, Functional equation

1. Ramanujan summation

1.1. Introduction

Ramanujan summation [1-4] is a fascinating concept developed by the Indian mathematician Srinivasa

Ramanujan in the early 20th century, which provides a framework for assigning values to certain divergent series. Traditionally, the convergence of a series is a prerequisite for its sum to be defined.

However, many series in math and physics diverge while showing structures that imply, they could still be summed in a wider sense. Ramanujan's insights push against the usual limits of analysis and present new methods to find finite sums from divergent series. The core of Ramanujan summation is that divergent series can be examined using techniques that look at their analytical continuation or visible patterns. This method not only improves our understanding of infinite sums but also leads to unexpected results, like connections between seemingly unrelated math ideas and relationships. Ramanujan summation affects multiple areas, such as number theory, combinatorics, and mathematical physics.

DOI: 10.54254/2753-8818/2025.31307

1.2. Use Bernoulli number to express $Mr(n)$

We start from a familiar series 1+2+3+4+...+100=5050, has a very interesting story from Gauss.

It means the formula of a finite consecutive $\sum_{k=1}^{n} k = \frac{1}{2} n (n+1)$, and we want to find more general and interesting sequence $\sum_{k=1}^{n} k^r$, now we denote this sequence as $\mathrm{Mr(n)} = \sum_{k=1}^{n} k^r$

And we can construct an exponential generating function (like a Maclaurin series form):

$$f(x;n) = \sum_{r=0}^{\infty} \frac{M_r(n)}{r!} x^r = \sum_{r=0}^{\infty} \frac{x^r}{r!} \sum_{k=1}^{n} k^r = \sum_{k=1}^{n} \sum_{r=0}^{\infty} \frac{(kx)^r}{r!}$$

$$= \sum_{k=1}^{n} e^{kx} = e^x \sum_{m=0}^{n-1} e^{mx}$$

$$= e^x \cdot \frac{e^{nx}-1}{e^x-1} = \frac{1-e^{nx}}{e^{-x}-1} \tag{1}$$

Generally, Bernoulli numbers [5-8] are a unique set of sequences that satisfy the following generating functions:

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k \Leftrightarrow \frac{-x}{e^{-x} - 1} = \sum_{k=0}^{\infty} \frac{(-1)^k B_k}{k!} x^k$$

So bring this to (1):

$$\begin{aligned} \frac{1-e^{nx}}{e^{-x}-1} &= \frac{e^{nx}-1}{x} \cdot \frac{-x}{e^{-x}-1} \\ &= \frac{e^{nx}-1}{x} \sum_{k=0}^{\infty} \frac{(-1)^k B_k}{k!} x^k \\ &= \left(\sum_{m=1}^{\infty} \frac{n^m}{m!} x^{m-1}\right) \left(\sum_{k=0}^{\infty} \frac{(-1)^k B_k}{k!} x^k\right) \end{aligned} \tag{2}$$

Lemma 1. Cauchy product:

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} a_m \, b_{n-m}\right)$$

So we can rearrange the summation in (2):

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$$\left(\sum_{m=1}^{\infty}\frac{n^m}{m!}x^{m-1}\right)\left(\sum_{k=0}^{\infty}\frac{(-1)^k B_k}{k!}x^k\right) = \left(\sum_{m=0}^{\infty}\frac{n^{m+1}}{(m+1)!}x^m\right)\left(\sum_{k=0}^{\infty}\frac{(-1)^k B_k}{k!}x^k\right)$$

$$\Rightarrow \sum_{r=0}^{\infty}\left(\sum_{m=0}^{r}\frac{(-1)^m B_m n^{r-m+1}}{m!\,(r-m+1)!}\right)x^r$$

$$= \sum_{r=0}^{\infty}\left(\sum_{m=0}^{r}\frac{r!\,(-1)^m B_m n^{r-m+1}}{m!\,(r-m+1)!}\right)\frac{x^r}{r!}$$

Then we get the formula of $M_r(n)$ expressed by Bernoulli numbers:

$$M_r(n) = \sum_{m=0}^{r}\frac{r!\,(-1)^m B_m n^{r-m+1}}{m!\,(r-m+1)!}$$

$$= \sum_{m=0}^{r}\frac{r!}{m!\,(r+1-m)!}(-1)^m B_m n^{r+1-m}$$

$$= \frac{1}{r+1}\sum_{m=0}^{r}\frac{(r+1)!}{m!\,[(r+1)-m]!}(-1)^m B_m n^{r+1-m}$$

$$= \frac{1}{r+1}\sum_{m=0}^{r}\binom{r+1}{m}(-1)^m B_m n^{r+1-m}$$

Therefore, the closed-end of the sum of powers of natural numbers can be expressed as:

$$M_r(n) = \sum_{k=1}^{n}k^r = \frac{1}{r+1}\sum_{m=0}^{r}\binom{r+1}{m}(-1)^m B_m n^{r+1-m} \quad (*)$$

1.3. Euler - Maclaurin formula and Ramanujan summation

For a polynomial function $f(x) = a_0 + a_1 x + \cdots + a_m x^m + \cdots = \sum_{r=0}^{\infty} a_r\, x^r$, to sum up both sides:

$$\sum_{k=1}^{n} f(k) = \sum_{k=1}^{n}\sum_{r=0}^{\infty} a_r\, k^r$$

$$= \sum_{r=0}^{\infty} a_r \sum_{k=1}^{n} k^r$$

$$= \sum_{r=0}^{\infty} a_r\, M_r(n)$$

Now, we can use (*) to do further calculation:

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$$\begin{aligned}\sum_{k=1}^{n} f(k) &= \sum_{r=0}^{\infty} a_r \cdot \frac{1}{r+1} \sum_{m=0}^{r} \binom{r+1}{m} (-1)^m B_m n^{r+1-m} \\ &= \sum_{r=0}^{\infty} \sum_{m=0}^{r} \frac{a_r}{r+1} \binom{r+1}{m} (-1)^m B_m n^{r+1-m} \\ &= \lim_{N\to\infty} \sum_{r=0}^{N} \sum_{m=0}^{r} \frac{a_r}{r+1} \binom{r+1}{m} (-1)^m B_m n^{r+1-m}\end{aligned} \tag{3}$$

Lemma 2.

$$\sum_{r=0}^{N} \sum_{m=0}^{r} b_{r,m} = \sum_{m=0}^{N} \sum_{r=m}^{N} b_{r,m}$$

So we can rearrange the summation in (3):

$$\begin{aligned}\sum_{k=1}^{n} f(k) &= \lim_{N\to\infty} \sum_{r=0}^{N} \sum_{m=0}^{r} \frac{a_r}{r+1} \binom{r+1}{m} (-1)^m B_m n^{r+1-m} \\ &= \lim_{N\to\infty} \sum_{m=0}^{N} \sum_{r=m}^{N} \frac{a_r}{r+1} \binom{r+1}{m} (-1)^m B_m n^{r+1-m} \\ &= \sum_{m=0}^{\infty} \sum_{r=m}^{\infty} \frac{a_r}{r+1} \binom{r+1}{m} (-1)^m B_m n^{r+1-m} \\ &= \sum_{m=0}^{\infty} (-1)^m B_m \sum_{r=m}^{\infty} \frac{a_r}{r+1} \binom{r+1}{m} n^{r+1-m} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m B_m}{m!} \sum_{r=m}^{\infty} \frac{r!}{(r+1-m)!} a_r n^{r+1-m} \\ &= B_0 \sum_{r=0}^{\infty} \frac{a_r n^{r+1}}{r+1} + \sum_{m=1}^{\infty} \frac{(-1)^m B_m}{m!} \sum_{r=m}^{\infty} \frac{r!}{(r+1-m)!} a_r n^{r+1-m}\end{aligned}$$

$$= B_0 \sum_{r=0}^{\infty} a_r \cdot \frac{n^{r+1}}{r+1} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1} B_{k+1}}{(k+1)!} \sum_{r=k+1}^{\infty} a_r \cdot \frac{r!\, n^{r-k}}{(r-k)!}$$

We know $\frac{n^{r+1}}{r+1} = \int_0^n x^r \, \mathrm{d}x$ and $\frac{r! n^{r-k}}{(r-k)!} = \frac{\mathrm{d}}{\mathrm{d}x} x^r \Big|_{x=n}$, therefore we can simplify this equation further:

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$$\sum_{k=1}^{n} f(k) = B_0 \sum_{r=0}^{\infty} a_r \int_0^n x^r \,\mathrm{d}x + \sum_{k=0}^{\infty} \frac{(-1)^{k+1} B_{k+1}}{(k+1)!} \sum_{r=k+1}^{\infty} a_r \frac{\mathrm{d}^k}{\mathrm{d}x^k}(x^r)\bigg|_{x=n}$$

$$= B_0 \int_0^n \sum_{r=0}^{\infty} a_r x^r \mathrm{d}x + \sum_{k=0}^{\infty} \frac{(-1)^{k+1} B_{k+1}}{(k+1)!} \cdot \left\{ \frac{\mathrm{d}^k}{\mathrm{d}x^k} \left[\sum_{r=k}^{\infty} a_r x^r \right]_{x=n} - a_k \right\}$$

$$= B_0 \int_0^n f(x) \mathrm{d}x + \sum_{k=0}^{\infty} \frac{(-1)^{k+1} B_{k+1}}{(k+1)!} \cdot \left\{ \frac{\mathrm{d}^k}{\mathrm{d}x^k} [f(x)]_{x=n} - \frac{\mathrm{d}^k}{\mathrm{d}x^k} \left[\sum_{r=0}^{k-1} a_r x^r \right]_{x=n} - \frac{\mathrm{d}^k}{\mathrm{d}x^k} [f(x)]_{x=0} \right\}$$

$$= \mathrm{B}_0 \int_0^n f(x) \mathrm{d}x + \sum_{k=0}^{\infty} \frac{(-1)^{k+1} B_{k+1}}{(k+1)!} \cdot \left[f^{(k)}(n) - f^{(k)}(0) \right]$$

$$= \mathrm{B}_0 \int_0^n f(x) \mathrm{d}x - B_1 [f(n) - f(0)] + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} B_{k+1}}{(k+1)!} \cdot \left[f^{(k)}(n) - f^{(k)}(0) \right]$$

$$= \int_0^n f(x) \mathrm{d}x + \frac{f(n) - f(0)}{2} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} B_{k+1}}{(k+1)!} \cdot \left[f^{(k)}(n) - f^{(k)}(0) \right]$$

$$= \int_0^n f(x) \mathrm{d}x + \frac{f(n) - f(0)}{2} + \sum_{k=1}^{\infty} \frac{(-1)^{2k} B_{2k}}{(2k)!} \left[f^{(2k-1)}(n) - f^{(2k-1)}(0) \right]$$

$$= \int_0^n f(x) \mathrm{d}x + \frac{f(n) - f(0)}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left[f^{(2k-1)}(n) - f^{(2k-1)}(0) \right]$$

Finally, we can get the formula for the summation of polynomial function:

$$\sum_{k=0}^{n} f(k) = \int_0^n f(x) \mathrm{d}x + \frac{f(n) + f(0)}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left[f^{(2k-1)}(n) - f^{(2k-1)}(0) \right]$$

And use the same way to express f(x) sum from 1 to n:

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x) \mathrm{d}x + \frac{f(n) + f(1)}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left[f^{(2k-1)}(n) - f^{(2k-1)}(1) \right] \quad (**)$$

So this is Euler - Maclaurin Formula [9,10].
Move (**) all terms related to n to the left-hand side:

$$\sum_{k=1}^{n} f(k) - \int_0^n f(x) \mathrm{d}x - \frac{f(n)}{2} - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(n)$$

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$$= \int_1^0 f(x)\mathrm{d}x + \frac{f(1)}{2} - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(1)$$

If the limit does not exist for n→+∞ on the left side of the equation, we treat the value on the right side of the equation as a divergent series $\sum_{k=1}^{n} f(k)$ as Ramanujan summation, denote as $\sum_{k=1}^{\mathrm{R}} f(k)$

$$\text{i.e.} \sum_{k=1}^{\mathrm{R}} f(k) = \int_1^0 f(x)\mathrm{d}x + \frac{f(1)}{2} - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(1)$$

1.4. Ramanujan summation method to give divergent series $\sum_{k=1}^{\infty} k$ assign a number

$$\sum_{k=1}^{R} k = \int_1^0 x\,\mathrm{d}x + \frac{1}{2} - \frac{B_2}{2} = -\frac{1}{2} + \frac{1}{2} - \frac{1}{12} = -\frac{1}{12}$$

2. Functional equation

2.1. Introduction

We know the definition of zeta function [11]: $\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}$, i.e. $\zeta(-1) = 1+2+3+\ldots = \sum_{k=1}^{R} k$.

But by the definition of zeta function k ∈ ℤ , so I want to use analogy from Gamma function [12-16] that extend the domain [17]. The functional equation [18] of the zeta function can be written as:

$$\Gamma(s/2)\zeta(s) = \pi^{s-\frac{1}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

The functional equation of the Riemann zeta function [11,19] provides a deeper connection between values of the zeta function at s and 1-s, expect s=1.The functional equation itself describes an extraordinary symmetry present in the zeta function and serves as a foundation for a wide range of results in number theory, which play pivotal roles in understanding prime distribution and density.

2.2. Functional equation of zeta function from gamma function

Recall $\Gamma(s) = \int_0^{\infty} \chi^{s-1} e^{-x} dx$ converges for all Re(s) >1 ∈ ℂ

substitute s→s/2 , x→㎡ π x ⇒ dx = ㎡ π dx , then

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$$\begin{aligned}\Gamma(s/2) &= \int_0^\infty \left(\mathrm{m}^2\pi x\right)^{s/2-1} e^{-\mathrm{m}^2\pi x}\mathrm{m}^2\pi dx \\ &= \int_0^\infty m^s\, \pi^{s/2}x^{s/2-1}e^{-m^2\pi x}dx \\ &= m^s\pi^{s/2}\int_0^\infty x^{s/2-1}\, e^{-m^2\pi x}dx \\ &\Rightarrow \pi^{-s/2}\Gamma(s/2)\frac{1}{m^s} = \int_0^\infty x^{s/2-1}\, e^{-m^2\pi x}dx \\ &\pi^{-s/2}\Gamma(s/2)\sum_{m=1}^{\infty}\frac{1}{m^s} = \int_0^\infty x^{s/2-1}\sum_{m=1}^{\infty} e^{-m^2\pi x}\, dx\end{aligned}$$

Denote $\psi(x)$=$\sum_{m=1}^{\infty} e^{-m^2\pi x}$, by poisson summation

$$\Rightarrow 2\psi(x)+1 = \sum_{-\infty}^{\infty} e^{-\mathrm{m}^2\pi x} = \frac{1}{\sqrt{x}}\sum_{-\infty}^{\infty} e^{-\mathrm{m}^2\pi\frac{1}{x}} = \frac{1}{\sqrt{x}}\left(2\psi\left(\frac{1}{x}\right)+1\right)$$

Since $\sum_{m=1}^{\infty} e^{-m^2\pi x}$ is converges absolutely, so we can interchange the notation of summation and integral:

$$\begin{aligned}\pi^{-s/2}\Gamma(s/2)\zeta(s) &= \int_0^\infty x^{s/2-1}\,\psi(x)dx \\ &= \int_0^1 x^{s/2-1}\,\psi(x)dx + \int_1^\infty x^{s/2-1}\,\psi(x)dx \\ &= \int_0^1 x^{s/2-1}\left(\frac{1}{2\sqrt{x}}\left(2\psi\left(\frac{1}{x}\right)+1\right)-\frac{1}{2}\right)dx + \int_1^\infty x^{s/2-1}\,\psi(x)dx \\ &= \int_0^1 x^{s/2-3/2}\,\psi\left(\frac{1}{x}\right)dx + \frac{1}{2}\int_0^1 x^{s/2-3/2}\,dx - \frac{1}{2}\int_0^1 x^{s/2-1}\,dx + \int_1^\infty x^{s/2-1}\,\psi(x)dx \\ &= \int_1^\infty x^{s/2-3/2}\,\psi\left(\frac{1}{x}\right)dx + \int_1^\infty x^{s/2-1}\,\psi(x)dx + \frac{1}{s(s-1)} \\ &= \int_1^\infty \left(x^{-s/2-1/2}+x^{s/2-1}\right)\psi(x)dx + \frac{1}{s(s-1)}\end{aligned}$$

We notice that if we change s to 1-s, the value of integral has not changed, so we can get the functional equation of zeta function:

$$\Rightarrow \Gamma(s/2)\zeta(s) = \pi^{s-\frac{1}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)\ (***)$$

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2.3. Zeta function method to give divergent series $\sum_{k=1}^{\infty} k = \zeta(-1)$

Put s = -1 to (***)

$$\text{i.e } \Gamma\left(-\frac{1}{2}\right)\zeta(-1) = \pi^{-\frac{3}{2}}\Gamma(1)\zeta(2)$$

since we know the properties of Γ(s) [20]: Γ(s) = Γ(s+1)/s, Γ(s+1) = sΓ(s) for all s >0

$$\Rightarrow \zeta(-1) = \frac{\pi^{-\frac{3}{2}}\Gamma(1)\zeta(2)}{\Gamma\left(-\frac{1}{2}\right)} = \frac{\pi^{-\frac{3}{2}}}{\pi^{\frac{1}{2}}} * 1 * \frac{\pi^2}{6} = -\frac{1}{12}$$

Then we use this different way to get a same result as Ramanujan summation method.

3. Conclusion

In this paper, we derive the summation formula of natural numbers by interpreting Gauss's discovery; By using Bernoulli numbers and their generating functions, the closed summation of natural numbers to powers is derived. By extending the summation formula of natural numbers to summation of power series, Euler-Maclaurin summation formula is obtained. Using the Ramanujan summation based on the Euler-MacLaurin formula, we once again explain why the sum of all natural numbers is set to negative one twelfth. And we can get the same result by zeta function. Since we use analogy by extension of domain of Gamma function. ζ(-1) = -1/12 is a result of extending the Riemann zeta function to negative integers using analytic continuation.

In conclusion, the study of Ramanujan summation, the Riemann zeta function, and divergent series reveals a rich tapestry of interconnections that significantly enhance our understanding of mathematical analysis. By extending the meaning of summation beyond traditional convergence, Ramanujan brought a new perspective and a set of tools that could be utilized to draw useful conclusions even from divergent series. A prime generator, the Riemann zeta function, is a linchpin in number theory, revealing deep connections amongst various fields of mathematics, but in particular when it comes to the distribution of prime numbers.

This we can do more by applying the Ramanujan summation techniques and the various properties of the zeta function and making the most use of the divergent sequence and therefore can applied to serve the whole branches, whether it is mathematical physics or complex analysis. Such exploration reinforces the significance of reexamining and reinterpreting classical mathematical constructs using modern methods. As investigations in this domain progress, we expect even more breakthroughs connecting disparate mathematical frameworks, leading to new methods for tackling outstanding questions.

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