Proceedings of CONF-CIAP 2026 Symposium: Applied Mathematics and Statistics
DOI: 10.54254/2753-8818/2026.CH31248

Simple Introduction to Eigenvalue and Similarity Form in Several Situations

Donglin Zeng

School of the Gifted Young, University of Science and Technology of China, Hefei, China
zdl2024@mail.ustc.edu.cn

Abstract. The eigenvalues and similarity theory of matrices have wide applications in various branches of mathematics. For a general matrix, a simple and usable standard form and an effective method to transform it into the standard form are required. Based on certain theoretical foundations, including some narration on solution space, linear independence of eigenvectors, elementary matrices, and Kronecker product, the process involves triangulation, diagonalization, Jordan standard form, and a convenient similarity classification method over general number fields. Some introductions and derivations are provided for similar standard forms, leading to the conditions for diagonalizability of general matrices and other standard forms for some situations when diagonalization is not possible. This also gives some deeper conceptions like determinant, divisor, and zero polynomial. For some special forms of matrices, there are stronger conclusions. Additionally, applications of related methods in problems are presented, such as matrix exponentiation, and in other aspects, like solving differential equations and linear sequence problems.

Keywords: Eigenvalues, Similarity, Diagonalization, Standard Form

1. Introduction

The concept of a matrix is derived from solving systems of linear equations or linear mappings, and matrix multiplication is defined through the composition of linear mappings. This has led to the rich content of matrix theory, including important concepts such as determinants, offsets, and similarities. Among them, as a special type of matrix product decomposition, it is mathematically applied to iterative mapping, matrix inversion, and solving systems of differential equations.

Simply put, the similarity transformation of a matrix can be understood as mapping the vector space to a new base representation under a certain linear transformation, and then mapping it back to the original base. The transformed B is expected to have a structure as concise as possible, such as diagonal matrices, because such matrices correspond to independent scaling in the coordinate direction. Expanding $A = PBP^{-1}$ will obtain an expression of the form $Ax = \lambda x$, which defines eigenvalues and eigenvectors. Eigenvectors are vectors that maintain their direction under the action of a matrix, while the corresponding eigenvalues characterize the scaling factor along that direction.

Generally, an n×n square matrix is similar to a diagonal matrix if and only if it has n eigenvectors. For matrices with n distinct eigenvalues, this is evident. For cases with repeated roots,

DOI: 10.54254/2753-8818/2026.CH31248

the existence of n eigenvectors can be determined by computing the rank of $\lambda I - A$[1].

The Cayley-Hamilton theorem $\varphi_A(A) = \prod_{i=1}^{n}(A-\lambda_i I) = O$, pointed out the relationship between eigenvalues and zeroing polynomials and minimum polynomials [2].

For more general non-diagonalizable situations, there is the Jordan standard form, which is similar to a tridiagonal matrix. It can be obtained through further elementary transformations after triangulation and quasi-diagonalization, or by decomposing the root subspace and cyclic subspace in linear space theory. By modulus offsetting of polynomial matrices $\lambda I - A$, similar standard forms can also be derived over specific number fields [3].

After importing these theoretical foundations, this article mainly introduces some definitions and conclusions of matrix similarity, lists common similarity standard forms and simple properties, and briefly introduces some applications of matrix similarity theory.

2. Theoretical foundations

For n×n square matrix A and B, there exists an invertible square matrix P, make $A = PBP^{-1}$, called A is similar to B . For n×n square matrix A, exists vector x and number λ , make $Ax = \lambda x$, called λ is eigenvalue and x is eigenvector.

For n×n square matrix A, if $rank(A) = m$, the solution space of $Ax = 0$ has $n - m$ basis vectors. (a) Eigenvalue λ make the matrix $\lambda I - A$ singular, so for each eigenvalue, there exists at least one eigenvector. (b) If A has n eigenvectors, the n eigenvectors are linearly independent. (c) Suppose E_{ij} is a matrix with 1 at i-th row and j-th column and 0 at others.

Define 3 types of elementary matrix as follow:

$$S_{ij} = I - (E_{ii} + E_{jj}) + (E_{ij} + E_{ji}), D_i(\lambda) = I + (\lambda - 1)E_{ii}, T_{ij}(\lambda) = I + \lambda E_{ij} \tag{1}$$

Easy to find that ${S_{ij}}^{-1} = S_{ij}, D_i(\lambda)^{-1} = D_i(\lambda^{-1}), T_{ij}(\lambda)^{-1} = T_{ij}(-\lambda)$.

So $S_{ij}AS_{ij}, D_i(\lambda)AD_i(\lambda^{-1}), T_{ij}(\lambda)AT_{ij}(-\lambda)$ are similar transforms [3]. Define the Kronecker product as follows:

$$A \otimes B = \begin{pmatrix} a_{ij}B \end{pmatrix} = \begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix} \tag{2}$$

For n×n square matrix A and B, this operation gives a $n^2 \times n^2$ square matrix.

Suppose A, B, X are n×n square matrix. Suppose

$$x = \begin{pmatrix} X_{11} & \dots & X_{n1} & X_{12} & \dots & X_{n2} & X_{1n} & \dots & X_{nn} \end{pmatrix} \tag{3}$$

Then $AX = (I \otimes A)x, XA = (A^T \otimes I)x$. This can transform matrix multiplication into the multiplication of matrices and vectors.

DOI: 10.54254/2753-8818/2026.CH31248

3. Deduction of similar standard forms

3.1. Triangulation and quasi-diagonalization

Suppose the eigenvalue is λ_1 . From (b), exists an eigenvector x_1 . There is a invertible P_1 , the first column is x_1 , make $AP_1 = P_1 \begin{pmatrix} \lambda_1 & * \\ 0 & * \end{pmatrix}$, and $*$ on lower right corner is $(n-1) \times (n-1)$ submatrix. From this, any matrix can be similar to an upper triangular square matrix, with n eigenvalues on the main diagonal [3].

Suppose in the upper triangular matrix B , B_{ii} and B_{jj} are different eigenvectors, then $T_{ij}(t)BT_{ij}(-t)$ make (i,j) element becomes 0 , where $t = \frac{B_{ij}}{B_{ii}-B_{jj}}$. From this, B can be further transformed into diagonal blocks, and the diagonal elements of each diagonal block have the same eigenvalues.

For the case where the eigenvalues are real numbers, if P_1 is taken as an orthogonal matrix, then A can be orthogonal and similar to an upper triangular matrix [4].

3.2. Diagonalization

If A has n different eigenvalues, from (b), A has n eigenvectors. Noted as $\lambda_1, \lambda_2, \ldots, \lambda_n,\ x_1, x_2, \ldots, x_n$. So

$$A(x_1 \quad x_2 \quad \cdots \quad x_n) = (x_1 \quad x_2 \quad \cdots \quad x_n) diag(\lambda_1 \quad \lambda_2 \quad \cdots \quad \lambda_n) \tag{4}$$

Suppose $P = (x_1 \quad x_2 \quad \cdots \quad x_{n_j}), D = diag(\lambda_1 \quad \lambda_2 \quad \cdots \quad \lambda_n)$, then $AP = PD$. From (c), P is invertible, so $A = PDP^{-1}$. If A has repeated eigenvalues, suppose λ_j has a multiplicity of n_j . If $rank(\lambda_j I - A) = n - n_j$, from (a), exist n_j eigenvectors correspond to λ_j . Noted as $x_1, x_2, \ldots, x_{nj}$. Then

$$A(x_1 \quad x_2 \quad \cdots \quad x_{n_j}) = (x_1 \quad x_2 \quad \cdots \quad x_{n_j}) \lambda_n I \tag{5}$$

If for each eigenvalue, there exists an eigenvector with a number equal to its multiplicity, A is diagonalizable.

3.3. Non-diagonalizable: Jordan standard form

If A is non-diagonalizable, A can be similar to the form $J = diag(J_{m_1}(\lambda_1), \cdots, J_{m_k}(\lambda_k))$, where $J_{m_k}(\lambda_k)$ is $m_k \times m_k$ Jordan block [5].

$$\begin{pmatrix} \lambda_k & 1 & & & \\ & \lambda_k & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_k & 1 \\ & & & & \lambda_k \end{pmatrix} \tag{6}$$

DOI: 10.54254/2753-8818/2026.CH31248

Easy to proof that $rank(J_{m_1}(\lambda_1)-\lambda_1 I)=m_1-1,\ \ rank(J_{m_1}(\lambda_1)-\lambda_1 I)^{m_1}=O$.

For eigenvalue λ_1, suppose the corresponding Jordan block is $A_1=diag(J_{m_1}(\lambda_1),\cdots,J_{m_k}(\lambda_1))$. Then $rank\left(A_1-\lambda_1 I\right)=\sum_{i=1}^{k}m_i-k$ .For $(A_1-\lambda_1 I)^2$, 1-order Jordan blocks in $(A_1-\lambda_1 I)$ are transformed to O, so $(A_1-\lambda_1 I)^2$ reduction in rank is the amount of Jordan blocks with at least 2 order compare to $(A_1-\lambda_1 I)$. Suppose there are a_k Jordan blocks with k-order, then $rank(A_1-\lambda_1 I)^{k-1}-rank(A_1-\lambda_1 I)^k$ is the amount of Jordan blocks with at least k order, as well as $\sum_{i\geq k}a_i$. Then

$$a_k=\sum_{i\geq k}a_i-\sum_{i\geq k+1}a_i$$

$$\begin{aligned}&=\left(rank(A_1-\lambda_1 I)^{k-1}-rank(A_1-\lambda_1 I)^k\right)\ \ =rank(A_1-\lambda_1 I)^{k-1}-2rank(A_1-\lambda_1 I)^k\\&-\left(rank(A_1-\lambda_1 I)^k-rank(A_1-\lambda_1 I)^{k+1}\right)\ \ +rank(A_1-\lambda_1 I)^{k+1}\end{aligned}\tag{7}$$

From this, the structure of the Jordan block can be inferred. Now giving the transition matrix P. For the situation of one Jordan block, suppose $J=J_m(\lambda),P=\begin{pmatrix}x_1 & x_2 & \cdots & x_m\end{pmatrix}$. Then $rank(A-\lambda I)^k=m-k$.

Set $k=1$, so x_1 is the only eigenvector corresponding to λ. First 2 columns of $AP=PJ$ are $Ax_1=\lambda x_1,\ Ax_2=x_1+\lambda x_2,\ (A-\lambda I)x_2=x_1$. Then $(A-\lambda I)^2x_1=(A-\lambda I)x_2=O$. As $(A-\lambda I)^2$ has a null space of 2 dimensions, and a basis is x_1, so x_2 is exist and unique. The remaining column vectors are similar [6].

For multiple Jordan block, as $(A-\lambda I)x_2=x_1$,the intersection space between $Ker(A-\lambda I)$ and $Im(A-\lambda I)$ can be calculated to choose x_1 in each Jordan block.

3.4. More general situations

For a real matrix with complex eigenvalues, it can be proved that any real matrix is similar to $diag(A_1,\ldots,A_k)$, where A_j are Jordan blocks or this form:

$$\begin{pmatrix}C & I_2 & & & \\ & C & I_2 & & \\ & & \ddots & \ddots & \\ & & & C & I_2\\ & & & & C\end{pmatrix},C=\begin{pmatrix}a_j & b_j\\ -b_j & a_j\end{pmatrix}\tag{8}$$

$a_j,\ b_j$ are corresponding to eigenvalue $a_j+ib_j(b_j\neq 0)$[3].

Suppose there is only one block. For complex eigenvalue $a+bi$ and complex eigenvector $x+yi$,

$$A\left(x+yi\right)=\left(ax-by\right)+\left(ay+bx\right)i,A\begin{pmatrix}x & y\end{pmatrix}=\begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}a & b\\ -b & a\end{pmatrix}\tag{9}$$

DOI: 10.54254/2753-8818/2026.CH31248

Suppose $u_i = (x_{i1} \quad \dots \quad x_{in} \quad y_{i1} \quad \dots \quad y_{in})^T$, u_1 is corresponding to real and imaginary parts of the eigenvalue. Using the definition of Kronecker product (2), note that

$$(I_2 \otimes A)u = (C^T \otimes I_n)u \tag{10}$$

Then

$$A(x_1 \quad y_1) = (x_1 \quad y_1)C, A(x_{k+1} \quad y_{k+1}) = (x_k \quad y_k) + (x_{k+1} \quad y_{k+1})C$$

$$(I_2 \otimes A)u_1 = (C^T \otimes I_n)u_1, (I_2 \otimes A)u_{k+1} = (C^T \otimes I_n)u_{k+1} + u_k \tag{11}$$

Suppose $M = I_2 \otimes A - C^T \otimes I_n = \begin{pmatrix} A - aI_n & bI_n \\ -bI_n & A - aI_n \end{pmatrix}$, so $Mu_1 = 0, \quad Mu_{k+1} = u_k$.

Similar to a real number, u_k can be calculated. More generally, it can be proven that the necessary and sufficient condition for A and B to be similar in a certain number field is the characteristic matrix $xI - A$ and $xI - B$ are modulus offset as polynomial matrices over the field of numbers [3]. This can be achieved by calculating the determinant, divisor, and invariant factors of the characteristic matrix.

3.5. Some special matrix

For a symmetric matrix A, as A is orthogonally similar to an upper triangular matrix, noted as $A = PBP^T$, then $A^T = PB^TP^T$, so B is diagonal, A is diagonalizable, and eigenvalues are all real [3].

For skew-symmetric matrix A, the same as above, B is skew-symmetric, so B has diagonal block of 0 or $\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$, so A's eigenvalues have a real part of 0 [7]. Suppose $A^k = B$, as A can be similar to an upper triangular matrix, easy to prove, B's eigenvalues are the k-th power of A's corresponding eigenvalue, and A's eigenvectors are also B's eigenvectors.

For orthogonal matrix A, as A is orthogonally similar to an upper triangular matrix, then $I = AA^T = PBB^TP^T, BB^T = I$. Calculate the diagonal elements of BB^T , easy to prove that $|B_{ii}| = 1$, which means the norm of A's eigenvalues are all 1.

4. Applications

4.1. Matrix power and elementary functions

Pick A's Jordan form (5). For a Jordan block,

DOI: 10.54254/2753-8818/2026.CH31248

$$J\left(\lambda\right)^k = \left(\lambda I + J\left(0\right)\right)^k = \sum_{i=0}^{k} J(0)^i C_k^i \lambda^{k-i} = \begin{pmatrix} \lambda^k & k\lambda^{k-1} & \cdots & & C_k^n \lambda^{k-n} \\ & \lambda^k & k\lambda^{k-1} & & \\ & & \ddots & \ddots & \vdots \\ & & & \lambda^k & k\lambda^{k-1} \\ & & & & \lambda^k \end{pmatrix} \quad (12)$$

So $A^k = PJP^{-1}PJP^{-1}\ldots PJP^{-1} = PJ^kP^{-1}$. Thus, the k-th power operation can be simplified as simple matrix multiplication.

For elementary functions, they can be transformed into a power operation by Taylor expansion

$$f(x) = \sum_{i=0}^{\infty} f^{(i)}(0)x^i \quad (13)$$

Thus, defining the elementary functions of the matrix [8].

4.2. Inverse of a matrix

From A's Jordan form, suppose the maximum order of Jordan block corresponding to eigenvalue λ_i is b_i , then $(A-\lambda I)^{b_i}$ make the diagonal block corresponding to λ_i all transformed to O, and b_i is the minimum power. So the zeroing polynomial $d_A = \prod_{i=1}^{m}(x-\lambda_i)^{b_i}$ can be determined. Suppose

$$d_A = x^n + \sum_{i=0}^{n-1} c_i x^i \quad (14)$$

Then

$$A^n + \sum_{i=1}^{n-1} c_i A^i + c_0 I = O, A\left(A^{n-1} + \sum_{i=1}^{n-1} c_i A^{i-1}\right) = -c_0 I$$

$$A^{-1} = -\frac{1}{c_0}\left(A^{n-1} + \sum_{i=1}^{n-1} c_i A^{i-1}\right) \quad (15)$$

As an invertible, λ_i is nonzero, c_0 as the product of several eigenvalues is also non-zero.

4.3. General formula for linear recursive sequence

For a linear recursive sequence $a_{n+k} = c_k a_{n+k-1} + \ldots + c_2 a_{n+1} + c_1 a_n$, suppose $x_n = (a_{n+k-1} \ \ \ldots \ \ a_n)^T$, then

DOI: 10.54254/2753-8818/2026.CH31248

$$x_{n+1} = \begin{pmatrix} c_k & c_{k-1} & \cdots & \cdots & c_2 & c_1 \\ 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & 1 & 0 \end{pmatrix} x_n \tag{16}$$

The matrix on the right of (16) is denoted as A. Notice that $det(\lambda I - A) = \sum_{i=0}^{k} c_i \lambda^i$, then the eigenvalues of A are the roots of a polynomial equation $\sum_{i=0}^{k} c_i \lambda^i = 0$. From $x_{n+1} = A^n x_1$ and given x_1 , the general term can be calculated [9, 10].

4.4. Systems of differential equations

Suppose $x_1, x_2, \cdots, x_n$ are differentiable function of t. Consider the differential equations

$$\frac{dx_i}{dt} = \sum_{j=0}^{n} c_{ij} x_j \tag{17}$$

Note that $A = (c_{ij}), x = (x_1 \quad \ldots \quad x_n)^T$, the equation can be denoted as $\frac{dx}{dt} = Ax$ .Speculated that the form of solution is

$$e^{Ax} = 1 + Ax + \frac{1}{2}(Ax)^2 + \ldots + \frac{1}{n!}(Ax)^n + \ldots \tag{18}$$

From $|A^k| = |A|^k$, this expression is convergence. Then transformed into the problem of matrix powers. For a nonlinear differential equation $\frac{dx_i}{dt} = f_i(x_1, x_2, \cdots, x_n)$, applying Taylor expansion (13) at a certain point, transforming it into a linear equation, and obtaining an approximate solution near this point [6].

5. Conclusion

The article mainly introduces some methods for processing matrices derived from the concepts of similarity and eigenvalues. A matrix with n eigenvectors can be similar to a diagonal matrix, while a general matrix can be triangulated and transformed into the Jordan standard form or other forms on a specific number field. For some special matrices, such as symmetric matrices, there are some simple processing methods.

Currently, the Jordan standard form can serve as the best solution for similar standard form problems in general number fields. The Jordan standard form, as the sum of an identity matrix and a nilpotent matrix, brings great convenience in complex operations. In practical applications, numerical calculations become the main difficulty when the theory is already relatively complete, especially when calculating the eigenvectors or transition matrix P, which requires repeatedly solving linear equation systems, and calculating the Jordan standard form by forcibly calculating matrix powers, etc. In the future, further research can be conducted in specific calculations, such as standardizing the solution of the transition matrix P.

DOI: 10.54254/2753-8818/2026.CH31248

References

[1] Xu, X. (2010). A review on diagonalization problems. Journal of Jiangsu Institute of Education.
[2] Zhang, H., Yin, H., & Mei, H. (2016). Several proofs of the Cayley–Hamilton theorem. Journal of Chuzhou University.
[3] Wang, X. (2025). Lecture notes on linear algebra. University of Science and Technology of China Press.
[4] Liu, A. (2016). Discussion on matrix orthogonal diagonalization and similarity diagonalization. Mathematics Teaching and Research.
[5] Yin, X., Yang, D., & Pan, M. (2021). Computation of Jordan standard form and similarity of matrices. College Mathematics.
[6] Liu, B. (2021). Ordinary differential equations. Peking University Press.
[7] Wei, Y., & Yin, L. (2021). On diagonalization of symmetric matrices in linear algebra of engineering mathematics. Journal of Binzhou University.
[8] Li, J., & Zhou, W. (2020). Application of diagonalizable matrices. Mathematics Learning and Research.
[9] Bai, H. (2022). Study on the application of diagonalizable square matrix. Heilongjiang Science.
[10] Zheng, H., & Xu, W. (2008). Application of matrix diagonalization method. Studies in College Mathematics.