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Study of an Advection–Diffusion Equation: A Variational Approach and the Zero-Diffusion Limit

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Abstract. The advection–diffusion equation with the homogeneous Dirichlet boundary conditions is studied in this paper. The analysis is carried out in a variational framework within a Sobolev space. We establish well-posedness of the equation by deriving explicit coercivity criteria that quantify how diffusion, reaction, and the divergence of the velocity field balance each other. Under a gradient vector field, we use a transformation that produces a second-order elliptic problem with an effective potential, thereby placing the problem in a coercive setting. We also establish continuity of the solution with respect to perturbations of the velocity and damping parameters, a property relevant to inverse modeling and numerical simulation. Finally, we derive a necessary and sufficient criterion for the zero-diffusion limit to exist in one-dimensional cases.

Keywords: Advection–Diffusion Model, Variational Formulation, Zero-Diffusion Limit

1. Introduction

Variational formulation recasts a PDE as the task of finding a function in a suitable Sobolev space that satisfies an identity against all test functions. A single integration by parts lowers the regularity required of solutions and cleanly encodes boundary conditions—Dirichlet as essential, Neumann/Robin as natural—while revealing an underlying energy structure. For linear problems, continuity together with coercivity ensures the existence and uniqueness of solution, with Poincaré's inequality providing quantitative control. The same framework underpins finite element methods, so analytic bounds transfer directly to computation (see also [1-5]).

Advection–diffusion models constitute a foundational framework for describing the transport of heat, solutes, and passive tracers across fluids and porous media. In steady settings, they encode a balance between directional transport driven by a velocity field and isotropic smoothing generated by diffusion. Let $\Omega \subset \mathbb{R}^n\ (n = 1, 2, 3)$, whose boundary is regular. Given a diffusion coefficient $\nu > 0$, a damping (reaction) coefficient $\alpha \geq 0$, a velocity field $\mathbf{V} \in C^1\left(\bar{\Omega}\right)^n$, and a source $f \in L^2(\Omega)$, the boundary-value problem we study is

$$-\nu\,\Delta u \;+\; \mathbf{V}\cdot\nabla u \;+\; \alpha\,u \;=\; f \quad \text{in}\Omega, \qquad u|_{\partial\Omega} = 0 \tag{1}$$

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Physically, u may represent temperature, concentration, or another scalar tracer transported by the flow $\mathbf{V}$ while being smoothed by diffusion and possibly attenuated by a linear reaction term. This model underlies a wide range of applications, from pollutant dispersion in environmental flows to heat conduction in heterogeneous media and the steady asymptotics of time-dependent convection–diffusion processes (see also [6-10]).

Our analysis is built upon a weak formulation posed in $H_0^1(\Omega)$. Specifically, it is characterized by a bilinear operator that combines a diffusion contribution, a convection term induced by the vector field V , and a reaction component weighted by α , together with a linear functional associated with the source term f .

The weak solution $u \in H_0^1(\Omega)$ is defined as the function satisfying the associated variational identity for all admissible test functions $v \in H_0^1(\Omega)$,

$$a(u, v) = L(v) \tag{2}$$

Continuity of a and L is straightforward, while coercivity of a is controlled by the interplay among diffusion coefficient, damping coefficient, and the divergence of $\mathbf{V}$. Combined with Poincaré’s inequality on $H_0^1(\Omega)$, this identity enables sharp coercivity criteria and, hence, well-posedness. Also, when $\mathbf{V}$ is a gradient field $\mathbf{V} = \nabla\phi$, the transformation $w = e^{-\phi/(2\nu)}u$ symmetrizes the operator and produces a second-order elliptic problem with an effective potential

$$\frac{|\mathbf{V}|^2}{4\nu} - \frac{1}{2}\operatorname{div}\mathbf{V} + \alpha, \tag{3}$$

which again fits squarely into the Lax–Milgram framework.

Moreover, we quantify continuity of the solution with respect to $(\mathbf{V}, \alpha)$ —a practically relevant sensitivity result for inverse problems and numerical simulation. We also analyze the existence of the zero-diffusion limit in one dimension, which has a clear physical interpretation: diffusion regularizes and selects a unique steady profile by smoothing thin inflow layers; when $\nu \to 0^+$, this regularization disappears, leaving a transport-dominated state whose existence requires appropriate compatibility between forcing, reaction, and boundary inflow.

We now present the key findings of this work.

Theorem 1.1. Consider a bounded open set $\Omega \subset \mathbb{R}^n$. Let $\nu > 0$, $\alpha \geq 0$, $V \in C^1\left(\bar{\Omega}\right)^n$, and $f \in L^2\left(\Omega\right)$. The following statements follow:

Divergence-free velocity field. If $\operatorname{div}\mathbf{V} = 0$ in Ω , there is a solution $u \in H_0^1(\Omega)$ for the boundary value problem (1.1).

Bounded divergence velocity field. Let $C_1 > 0$ denote the optimal constant in Poincaré's inequality. If

$$\max_{\mathbf{x} \in \bar{\Omega}} \operatorname{div}\mathbf{V}(\mathbf{x}) < 2\alpha + \frac{2\nu}{C_1}, \tag{4}$$

Under the given assumptions, there exists exactly one solution to the boundary value problem (1.1), which is that $u \in H_0^1(\Omega)$.

Gradient vector fields. Let $C_1 > 0$ denote the constant appearing in Poincaré’s inequality, and $B := \max_{\mathbf{x} \in \bar{\Omega}} |\mathbf{V}(\mathbf{x})|^2$. Assume that $\alpha > 0$, or $\alpha = 0$ and $\nu > \frac{\sqrt{C_1 B}}{2}$. If $\mathbf{V} = \nabla\phi$ with

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$\phi \in C^2\left(\bar{\Omega}\right)$,

Under the given assumptions, there exists exactly one solution to the boundary value problem (1.1), which is that $u \in H_0^1\left(\Omega\right)$

Continuity with respect to $\left(\mathbf{V}, \alpha\right)$. Let $\operatorname{div}\mathbf{V} = 0$. Fix a diffusion coefficient $\nu > 0$. The mapping

$$\left(\mathbf{V}, \alpha\right) \mapsto u_{\nu, \mathbf{V}, \alpha}, \tag{5}$$

which acts from $C^1\left(\bar{\Omega}\right)^n \times \mathbb{R}_{\geq 0}$ into $H_0^1\left(\Omega\right)$, is continuous.

Zero-diffusion limit. Let $\Omega = (0,1)$ and $\alpha = 0$. Assume that $\mathbf{V}$ is a constant (divergence free, f is positive with bounded derivative on (0,1). Then the solution u_ν converges in $H_0^1(0,1)$ as $\nu \to 0^+$ when

$$\int_0^1 f\left(x\right) dx \ = \ 0 \quad \text{and} \quad \mathbf{V} \neq 0 \tag{6}$$

2. Preliminaries

In this section, we discuss integrations on open sets, and from time to time we have to deal with terms that involves integrations on boundaries of the sets; in order that the integration makes sense, the boundaries have to be regular enough. Therefore, for simplicity's sake, we assume all open sets mentioned in this paper are regular enough so that some formulas, the most important of which is Green's formula, can be applied to simplify certain results. We first provide the definition of regularity of an open set.

Definition 2.1. Ω has a boundary of class C^k , with an integer $k \geq 1$, if its boundary can be locally flattened by a finite family of coordinate transformations.

More precisely, there exists a finite open covering $\left\{\omega_i\right\}_{i=0}^{I}$ of a neighborhood of $\partial\Omega$ such that, for each index $i \in \{1, \ldots, I\}$, one can construct a bijective mapping ϕ_i: $\omega_i \longrightarrow Q$, and $Q \subset \mathbb{R}^N$ denotes a reference cube, and both ϕ_i and its inverse are of class C^k .

Under this transformation, the domain Ω is locally mapped onto a half-cube, while its boundary corresponds to the flat portion of Q . In particular, the image of $\omega_i \cap \Omega$ lies in the region where the last coordinate is positive, whereas $\omega_i \cap \partial\Omega$ is mapped onto the hyperplane where this coordinate vanishes.

2.1. Square-integrable functions and $\mathbf{L}^2\left(\boldsymbol{\Omega}\right)$

We then provide the definition of $L^2\left(\Omega\right)$.

Definition 2.2. For an open domain $\Omega \subset \mathbb{R}^N$ with Lebesgue measure, $L^2\left(\Omega\right)$ is the space of square-integrable functions on Ω .

Throughout this work, $\mathrm{L}^2\left(\Omega\right)$ is endowed with its standard inner product and the associated norm, which together turn it into a Hilbert space.

DOI: 10.54254/2753-8818/2026.CH31164

2.2. $\mathbf{H}^1(\boldsymbol{\Omega})$ and $\mathbf{H}_0^1(\boldsymbol{\Omega})$

Definition 2.3. We denote by $H^1(\Omega)$ the Sobolev space consisting of square-integrable functions whose first-order weak derivatives also belong to $L^2(\Omega)$.

The space $\mathrm{H}^1(\Omega)$ is equipped with its canonical inner product and the associated norm, which make it a Hilbert space. In what follows, this structure will be used without further comment.

Considering boundary conditions, we find it more useful sometimes to discuss a subset of $H^1(\Omega)$. Therefore, we first define below several important spaces on which $H_0^1(\Omega)$ is defined, and provides the complete defenition of it.

Definition 2.4. We adopt the standard notation $C(\Omega)$ and $C\left(\bar{\Omega}\right)$ for spaces of continuous functions defined on Ω and on its closure, respectively.

For an integer $\mathrm{k} \geq 1$, the spaces $\mathrm{C}^{\mathrm{k}}(\Omega)$ and $\mathrm{C}^{\mathrm{k}}\left(\bar{\Omega}\right)$ consist of functions whose partial derivatives up to order k are continuous in Ω and admit continuous extensions to $\bar{\Omega}$. The notation $\mathrm{C}^{\infty}(\Omega)$ and $\mathrm{C}^{\infty}\left(\bar{\Omega}\right)$ is used in the usual sense for infinitely differentiable functions.

Definition 2.5. $C_c^{\infty}(\Omega)$ is the space of smooth functions with compact support. $H_0^1(\Omega)$ is defined as the closure of $C_c^{\infty}(\Omega)$ with respect to the $H^1(\Omega)$ -norm.

The definition may seem rather confusing. To help understanding, we provide a few corollaries that provide an alternative definition, along with a norm defined within the space $H_0^1(\Omega)$.

Corollary 2.6. For an open set $\Omega \subset \mathbb{R}^N$, it has at least one bounded coordinate direction. Then the semi-norm

$$|v|_{H_0^1(\Omega)} = \sqrt{\int_{\Omega} |\nabla v(\mathbf{x})|^2 d\mathbf{x}} \tag{7}$$

is a norm on $H_0^1(\Omega)$, which is the same as the usual norm on $H^1(\Omega)$.

Corollary 2.7. Ω has a C^1 boundary. Then the space $H_0^1(\Omega)$ can be identified with the class of functions in $H^1(\Omega)$ whose traces vanish on $\partial\Omega$.

This result provides an equivalent characterization of $\mathrm{H}_0^1(\Omega)$: functions belonging to this space satisfy homogeneous boundary conditions in the sense of traces. This property proves extremely valuable in dealing with boundary conditions.

2.3. Green's formula and Poincaré's inequality

After introducing the space, we are especially dealing with (namely, the space $H_0^1(\Omega)$), we oftentimes find it not enough to establish a variational formulation associated with the partial differential equation. Here, we introduce Green's formula, a few corollaries of which are vital tools in transforming the equation to yield a preferable form.

Theorem 2.8. (Green's formula). Ω has a C^1 boundary, and let $w \in H^1(\Omega)$ be a function whose support is compactly contained in $\bar{\Omega}$. Then it verifies Green's formula

$$\int_{\Omega} \frac{\partial w}{\partial x_i}(\mathbf{x}) d\mathbf{x} = \int_{\partial\Omega} w(\mathbf{x}) n_i(\mathbf{x}) ds. \tag{8}$$

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Theorem 2.8 has a few corollaries that are direct consequences of the original Green's formula. We introduce two corollaries, one of which is the integration by parts formula.

Corollary 2.9. Ω has a C^1 boundary, and let $u, v \in H^1(\Omega)$ be functions with compact support in $\bar{\Omega}$.

Then the standard integration-by-parts formula holds in the weak sense, allowing the derivative to be transferred from one function to the other, together with a boundary contribution involving the outward unit normal vector.

Proof. The result follows directly from Theorem 2.8 by applying it to the product uv .

Corollary 2.10. Consider a regular C^1 open set Ω, with $u \in C^2(\Omega)$ and $v \in C^1(\Omega)$, each having bounded support in Ω.

Then Green's identity holds, relating the Laplacian of u tested against v to the gradient interaction between u and v , together with a boundary term involving the normal derivative of u .

Proof. The result is obtained by applying Corollary 2.9 componentwise to the partial derivatives of u and summing over all spatial directions.

Corollary 2.11. Consider a regular C^1 open set Ω, assume that $u \in C^2(\Omega)$ and that v is a C^1 vector field with compact support in Ω.

Then the divergence theorem holds in integral form: the integral of the scalar product between ∇u and v over Ω can be rewritten as the integral of u against the divergence of v, up to a boundary contribution involving the outward normal vector.

Proof. This identity follows by applying Corollary 2.9 to each component of v and summing over the spatial dimension.

Following, we provide Poincaré's inequality, which is crucial to verifying the coercivity (ellipticity) of a certain variational formulation.

Proposition 2.12. Ω is a set bounded in at least one coordinate direction and there is a positive constant C_1 such that every function $v \in H_0^1(\Omega)$ satisfies the Poincaré-type estimate

$$\int_\Omega v(\mathbf{x})^2 d\mathbf{x} \leq C_1 \int_\Omega |\nabla v(\mathbf{x})|^2 d\mathbf{x}. \tag{9}$$

2.4. Lax–Milgram Theorem

The Lax-Milgram Theorem provides a fundamental framework for analyzing variational problems in partial differential equations. Through verification of key properties in the variational formulation, this theorem establishes both uniqueness and existence of solutions to the class of PDEs. Below we present its precise mathematical framework, required conditions, and main conclusions.

Given a bilinear mapping $a(\cdot,\cdot)$ defined on the Hilbert space V and a linear functional L acting on V, the variational problem consists in finding an element $u \in V$ such that the associated weak identity holds when tested against arbitrary elements of V.

The Lax–Milgram Theorem requires four conditions: Linearity, Bilinearity, Continuity, and Coercivity (Ellipticity).

Theorem 2.13. Consider a continuous bilinear form $a : V \times V \to \mathbb{R}$ that is also coercive, and linear form $L : V \to \mathbb{R}$. Then, under these conditions, the variational formulation $L(v) = a(u, v)$ has only one solution.

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3. Analysis of the advection–diffusion equation

3.1. General overview

The advection–diffusion equation describes how a physical quantity, such as heat or concentration, experiences two processes, advection and diffusion, and changes by time as a result. Here, we examine the advection–diffusion model and analyze a few conditions, under which the equation admits a unique solution. Consider the model in $\mathbb{R}^n$ (where $n = 1, 2, 3$ in practice). In dimension when n is three, then $\mathbf{x} = (x_1, x_2, x_3)$. Ω has regular boundary $\partial\Omega$ and outward normal $\mathbf{n}$. $\nu > 0$ is the diffusion coefficient and $\alpha \geq 0$ is the damping parameter. Consider a velocity field $V : \mathbb{R}^n \to \mathbb{R}^n$ that is continuously differentiable on Ω including its boundary, so that $V \in C^1\left(\bar{\Omega}\right)^n$. Moreover, we suppose that $f \in L^2(\Omega)$.

We provide the advection–diffusion equation as following:

$$-\nu\Delta u + \mathbf{V} \cdot \nabla u + \alpha u = f, \quad \mathbf{x} \in \Omega; \tag{10}$$

together with

$$u = 0, \quad \mathbf{x} \in \partial\Omega. \tag{11}$$

The equation is tested against an arbitrary function $v \in H_0^1(\Omega)$. Using Green's identity and the vanishing trace of v on the boundary, the diffusion term can be rewritten in gradient form, while no boundary contribution arises.

As a result, the formulation of the advection–diffusion problem reads: find $u \in H_0^1(\Omega)$,

$$\nu \int_\Omega \nabla u \cdot \nabla v d\mathbf{x} + \int_\Omega (\mathbf{V} \cdot \nabla u) v d\mathbf{x} + \alpha \int_\Omega u v d\mathbf{x} = \int_\Omega f v d\mathbf{x}, \ \forall v \in H_0^1(\Omega) \tag{12}$$

For notational convenience, the dependence on the spatial variable will be omitted in the sequel. Introducing $a(\cdot,\cdot)$ and $L(\cdot)$ in the standard way, the above is also as follows:

$$a(u, v) = L(v), \tag{13}$$

Having established a variational formulation, what lefts to be shown is the conditions under which the Lax–Milgram Theorem 2.13 can be applied. Some of conditions are clearly true: due to the properties of integral, we find the linearity of $L(\cdot)$ and the bilinearity of $a(\cdot,\cdot)$ are trivial. The continuity of $L(\cdot)$ is also simple to verify. Moreover, we can easily verify the continuity of $a(\cdot,\cdot)$ using the following proposition.

Proposition 3.1. For a vector field $\mathbf{V} \in C^1\left(\bar{\Omega}\right)^n$ and function $u \in H_0^1(\Omega)$, we will get

$$\|\mathbf{V} \cdot \nabla u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)} \tag{14}$$

Proof. Since $V \in C^1\left(\bar{\Omega}\right)^n$, it is bounded on Ω , and hence belongs to $L^\infty(\Omega)^n$. The result follows directly from the pointwise estimate

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$$\left|\mathbf{V}\left(\mathrm{x}\right)\cdot\nabla u\left(x\right)\right|\leq\|\mathbf{V}\|_{L^{\infty}(\Omega)}\left|\nabla u\left(x\right)\right| \tag{15}$$

together with integration over Ω . □

As a consequence of this estimate, both the linear functional $L(\cdot)$ and the bilinear form $a(\cdot,\cdot)$ defined in the variational formulation are continuous on $H_0^1(\Omega)$.

Lemma 3.2. The linear functional $L(\cdot)$ and the bilinear form $a(\cdot,\cdot)$ associated with the variational formulation are continuous on $H_0^1(\Omega)$.

Proof. We first consider the linear functional L . By the Cauchy–Schwarz inequality and Poincaré's inequality, there exists

$$|L\left(v\right)|\leq C\,||\,v\,||_{\,H_0^1(\Omega)}, \tag{16}$$

which proves the continuity of $L(\cdot)$.

We next turn to $a(\cdot,\cdot)$. The diffusion term is bounded by the product of the $H_0^1(\Omega)$ -seminorms of its arguments. The convection term is controlled using the boundedness of the vector field V , together with Poincaré's inequality. Finally, the reaction term is estimated again by means of the Cauchy–Schwarz and Poincaré inequalities.

Collecting these bounds, one finds that

$$|a\left(u,v\right)|\leq C\,||\,u\,||_{\,H_0^1(\Omega)}\,||\,v\,||_{\,H_0^1(\Omega)}, \tag{17}$$

which establishes the continuity of $a(\cdot,\cdot)$. □

Therefore,

$$a\left(u,v\right)=a_1+a_2+a_3\leq\left(\nu+C_1C+\alpha C'\right)||\,u\,||_{\,H_0^1(\Omega)}\,||\,v\,||_{\,H_0^1(\Omega)}. \tag{18}$$

Since all of $\nu, C_1, C, \alpha,$ and C' are either positive or non-negative, we have verified the continuity of $a\left(\cdot,\cdot\right)$ as well, which completes our proof.

However, one of the conditions of Lax Milgram Theorem 2.13 is not always satisfied; especially, the coercivity of $a\left(\cdot,\cdot\right)$ requires some additional conditions to be satisfied. Thus, we now turn to an examination of solution existence and uniqueness for the variational formulation under several additional constraints.

3.2. Solution existence and uniqueness for a divergence-free velocity field

The situation in which the velocity field has zero divergence is examined first. In other words, $\mathrm{div}\left(\mathbf{V}\left(\mathbf{x}\right)\right)=0$. Recall the definition of divergence as following.

Definition 3.3. For a vector field $\mathbf{V}:\mathbb{R}^n\rightarrow\mathbb{R}^n$, define its divergence ($\mathrm{div}\left(\mathbf{V}\left(\mathbf{x}\right)\right)$ or, in short, $\mathrm{div}\mathbf{V}$) as

$$\mathrm{div}\left(\mathbf{V}\left(\mathbf{x}\right)\right)=\sum_{i=1}^{n}\frac{\partial V_i}{\partial x_i}\left(\mathbf{x}\right), \tag{19}$$

We introduce a few lemmas that would be useful later on when proving the coercivity of $a\left(\cdot,\cdot\right)$.

Lemma 3.4. Consider a vector field V mapping $\mathbb{R}^n$ into itself., $v:\mathbb{R}^n\rightarrow\mathbb{R}$ a function. Then

DOI: 10.54254/2753-8818/2026.CH31164

$$\operatorname{div}(v\mathbf{V}) = v\operatorname{div}\mathbf{V} + \mathbf{V} \cdot \nabla v. \tag{20}$$

Proof. By definition of divergence,

$$\operatorname{div}(v\mathbf{V}) = \sum_{i=1}^{n} \frac{\partial(vV_i)}{\partial x_i} = \sum_{i=1}^{n} \left(\frac{\partial v}{\partial x_i} V_i + v\frac{\partial V_i}{\partial x_i}\right) = \nabla v \cdot \mathbf{V} + v\operatorname{div}\mathbf{V} \tag{21}$$

as desired.

Lemma 3.5. Consider a vector field V mapping $\mathbb{R}^n$ into itself., $v : \mathbb{R}^n \to \mathbb{R}$ a function in $H_0^1(\Omega)$. Then

$$\int_\Omega (\mathbf{V} \cdot \nabla v)v d\mathbf{x} = \int_\Omega \nabla v \cdot (v\mathbf{V}) d\mathbf{x} = -\frac{1}{2}\int_\Omega v^2 \operatorname{div}\mathbf{V} d\mathbf{x}. \tag{22}$$

Proof. The first equality is trivial. Applying Corollary 2.11 to v and $v\mathbf{V}$ yields

$$\int_\Omega \nabla v \cdot (v\mathbf{V}) d\mathbf{x} = -\int_\Omega v\operatorname{div}(v\mathbf{V}) d\mathbf{x} + \int_{\partial\Omega} v((v\mathbf{V}) \cdot \mathbf{n}) d\mathbf{x} = -\int_\Omega v\operatorname{div}(v\mathbf{V}) d\mathbf{x}, \tag{23}$$

since $v = 0$ on $\partial\Omega$. Therefore, further apply Lemma 3.4 to the right,

$$\begin{aligned} \int_\Omega \nabla v \cdot (v\mathbf{V}) d\mathbf{x} &= -\int_\Omega v\operatorname{div}(v\mathbf{V}) d\mathbf{x} \\ &= -\int_\Omega v(v\operatorname{div}\mathbf{V} + \mathbf{V} \cdot \nabla v) d\mathbf{x} \\ &= -\int_\Omega v^2 \operatorname{div}\mathbf{V} d\mathbf{x} - \int_\Omega (v\mathbf{V}) \cdot \nabla v d\mathbf{x}. \end{aligned} \tag{24}$$

We can find that the term on the right happens to be equivalent to the left; hence, moving the term on the right to the left, then dividing both sides by 2 yields

$$\int_\Omega \nabla v \cdot (v\mathbf{V}) d\mathbf{x} = -\frac{1}{2}\int_\Omega v^2 \operatorname{div}\mathbf{V} d\mathbf{x} \tag{25}$$

as desired.

We proceed to show the following.

Theorem 3.6. In a divergence-free velocity field $(\operatorname{div}\mathbf{V} = 0)$, there is a unique solution in $H_0^1(\Omega)$ to the advection–diffusion equation (10).

Proof. We aim to show the coercivity of $a(\cdot,\cdot)$ in the variational formulation associated with the advection–diffusion equation:

$$a(v,v) = \nu\int_\Omega |\nabla v|^2 d\mathbf{x} + \int_\Omega (\mathbf{V} \cdot \nabla v)v d\mathbf{x} + \alpha\int_\Omega v^2 d\mathbf{x}. \tag{26}$$

Since $\operatorname{div}\mathbf{V} = 0$, we can get that the term on the right immediately vanishes by Lemma 3.5. We can thus get

$$\begin{aligned} a(v,v) &= \nu\int_\Omega |\nabla v|^2 d\mathbf{x} + \alpha\int_\Omega v^2 d\mathbf{x} \\ &\geq \nu \| v \|_{H_0^1(\Omega)}^2, \end{aligned} \tag{27}$$

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since $\|v\|_{L^2(\Omega)}$ is non-negative. Therefore, $a(\cdot,\cdot)$ is coercive; combined with the other conditions, which we have previously verified, the Lax–Milgram Theorem 2.13 is applied to the advection–diffusion equation under the condition of a divergence-free velocity field.

3.3. Solution existence and uniqueness for a velocity field having bounded divergence

We then assume that the velocity field, while not necessarily divergence-free, is bounded. In this section, we aim to find a range of $\mathrm{div}\mathbf{V}$ that guarantees the coercivity of $a(\cdot,\cdot)$ and therefore guarantees only one solution to the advection–diffusion equation.

Theorem 3.7. Only one solution to the advection–diffusion (10) with Dirichlet boundary conditions exists provided that

$$\max_{\mathbf{x}\in\bar{\Omega}}\mathrm{div}\mathbf{V}(\mathbf{x}) < 2\alpha + \frac{2\nu}{C_1}, \tag{28}$$

Applying Lemma 3.5, we can get

$$a(v,v) = \nu\int_\Omega |\nabla v|^2 d\mathbf{x} - \frac{1}{2}\int_\Omega v^2 \mathrm{div}\mathbf{V} d\mathbf{x} + \alpha\int_\Omega v^2 d\mathbf{x}. \tag{29}$$

To begin with, if $\mathrm{div}\mathbf{V} < 0$, then the second term is positive; as a result, $a(v,v) \geq \nu\|v\|_{H_0^1(\Omega)}$ still holds, guaranteeing the coercivity. This means that the infimum of the range of $\mathrm{div}\mathbf{V}$ that guarantees the coercivity is $-\infty$. What is left to be found, therefore, is a maximum (or, failing that, a supremum) for $\mathrm{div}\mathbf{V}$. In this case, we further simplify the above equation to

$$a(v,v) = \nu\int_\Omega |\nabla v|^2 d\mathbf{x} + \int_\Omega v^2\left(\alpha - \frac{1}{2}\mathrm{div}\mathbf{V}\right)d\mathbf{x}. \tag{30}$$

Denote by

$$C = \max_{\mathbf{x}\in\bar{\Omega}}\mathrm{div}\mathbf{V}(\mathbf{x}) \tag{31}$$

the maximum we aim to find. If $C \leq 2\alpha$, then $\alpha - \frac{1}{2}\mathrm{div}\mathbf{V} \geq 0$ for all $\mathbf{x}\in\Omega$, so $a(v,v) \geq \nu\|v\|_{H_0^1(\Omega)}$ holds as well, guaranteeing coercivity. Consider, moreover, the case in which $C > 2\alpha$. Then

$$\begin{aligned} a(v,v) &\geq \nu\int_\Omega |\nabla v|^2 d\mathbf{x} + \left(\alpha - \frac{1}{2}C\right)\int_\Omega v^2 d\mathbf{x} \\ &\geq \nu\int_\Omega |\nabla v|^2 d\mathbf{x} + \left(\alpha - \frac{1}{2}C\right)C_1\int_\Omega |\nabla v|^2 d\mathbf{x} \\ &= \left(\nu + \left(\alpha - \frac{1}{2}C\right)C_1\right)\int_\Omega |\nabla v|^2 d\mathbf{x}, \end{aligned} \tag{32}$$

where the second inequality is satisfied by Poincaré's Inequality (21). If $\nu + \left(\alpha - \frac{1}{2}C\right)C_1 > 0$ or, equivalently, $C < 2\alpha + \frac{2\nu}{C_1}$, then the coercivity is satisfied. This provides a supremum for $\mathrm{div}\mathbf{V}$.

In conclusion, if

$$\max_{\mathbf{x}\in\bar{\Omega}}\mathrm{div}\mathbf{V}(\mathbf{x}) < 2\alpha + \frac{2\nu}{C_1}, \tag{33}$$

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we can verify the coercivity of $a(\cdot,\cdot)$ before. Therefore, equation (10) admits a unique solution.

Remark 3.8. It should be noted that Theorem 3.7 does not imply that a velocity field with divergence outside this range cannot have a unique solution; the condition $max_{x\in\bar{\Omega}} divV(x) < 2\alpha + 2\nu/C_1$ is merely a sufficient condition for the one unique solution to the equation.

3.4. Solution existence and uniqueness for a gradient vector field

In this special situation, the velocity is the gradient of a prescribed scalar potential $\phi \in C^2\left(\bar{\Omega}\right)$, hence $\mathbf{V} = \nabla\phi$. We set $\lambda(\mathbf{x}) = \frac{\phi(\mathbf{x})}{2\nu}$ and define $w(\mathbf{x}) = e^{-\lambda(\mathbf{x})}u(\mathbf{x})$, where $u(\mathbf{x})$ is solution to advection–diffusion (10). Since $u = 0,\ \mathbf{x} \in \partial\Omega$, we also have $w = 0,\ \mathbf{x} \in \partial\Omega$. To investigate the uniqueness and existence of a solution when $\mathbf{V} = \nabla\phi$, we will first prove the following.

Lemma 3.9. $w \in X$ is a solution of

$$-\nu\Delta w + \left(\frac{1}{4\nu}|\mathbf{V}|^2 - \frac{1}{2}\mathrm{div}\mathbf{V} + \alpha\right)w = g,\ \ g = e^{-\lambda}f, \tag{34}$$

where ν is the diffusion coefficient and α is the damping parameter in the advection–diffusion equation (10).

Proof. We first calculate ∇w and Δw. By product rule, we can get

$$\nabla w = e^{-\lambda}\nabla u - ue^{-\lambda}\nabla\lambda = e^{-\lambda}(\nabla u - u\nabla\lambda). \tag{35}$$

Then we use ∇w to obtain Δw by noticing that

$$\Delta w = \mathrm{div}(\nabla w) = \mathrm{div}\left(e^{-\lambda}(\nabla u - u\nabla\lambda)\right). \tag{36}$$

Applying Lemma 3.4 to $\mathrm{div}(u\nabla\lambda)$ gives

$$\mathrm{div}(u\nabla\lambda) = \nabla u\cdot\nabla\lambda + u\Delta\lambda. \tag{37}$$

Therefore we have

$$\begin{aligned}\Delta w &= e^{-\lambda}\Delta u - e^{-\lambda}(\nabla u\cdot\nabla\lambda + u\Delta\lambda) - e^{-\lambda}\nabla u\cdot\nabla\lambda + e^{-\lambda}u|\nabla\lambda|^2 \\ &= e^{-\lambda}\Delta u - 2e^{-\lambda}\nabla u\cdot\nabla\lambda - e^{-\lambda}u\Delta\lambda + e^{-\lambda}u|\nabla\lambda|^2.\end{aligned} \tag{38}$$

From equation (10) we can derive

$$\Delta u = \frac{\mathbf{V}\nabla u + \alpha u - f}{\nu}. \tag{39}$$

Also, from $\lambda = \frac{\phi}{2\nu}$ and $V = \nabla\phi$, we have

$$\nabla\lambda = \frac{\mathbf{V}}{2\nu},\ \Delta\lambda = \frac{\mathrm{div}\mathbf{V}}{2\nu}. \tag{40}$$

Substituting Δu, $\nabla\lambda$ and $\Delta\lambda$ into the expression of Δw gives

DOI: 10.54254/2753-8818/2026.CH31164

$$\begin{aligned} \Delta w &= e^{-\lambda}\left(\frac{\mathbf{V}\nabla u+\alpha u-f}{\nu}\right)-2e^{-\lambda}\nabla u\cdot\left(\frac{\mathbf{V}}{2\nu}\right)-e^{-\lambda}u\left(\frac{\operatorname{div}\mathbf{V}}{2\nu}\right)+e^{-\lambda}u\left(\frac{|\mathbf{V}|^2}{4\nu^2}\right) \\ &= \frac{e^{-\lambda}\alpha u}{\nu}-\frac{e^{-\lambda}f}{\nu}-\frac{e^{-\lambda}u}{2}\operatorname{div}\mathbf{V}-\frac{e^{-\lambda}u|\mathbf{V}|^2}{4\nu}. \end{aligned} \tag{41}$$

Multiplying $-\nu$ to both sides gives

$$-\nu\Delta w=-\alpha w+e^{-\lambda}f+w\cdot\frac{\operatorname{div}\mathbf{V}}{2}-w\cdot\frac{|\mathbf{V}|^2}{4\nu}. \tag{42}$$

Rearranging the terms gives

$$-\nu\Delta w+\left(\frac{1}{4\nu}|\mathbf{V}|^2-\frac{1}{2}\operatorname{div}\mathbf{V}+\alpha\right)w=e^{-\lambda}f. \tag{43}$$

And since we define $e^{-\lambda}f$ as g, we complete the proof.

Now we declare $w(x)\in H_0^1(\Omega)$ for the following reason: given that $w(x)=e^{-\lambda(x)}u(x)$, we have derived

$$\nabla w = e^{-\lambda}\nabla u-Ve^{-\lambda}u. \tag{44}$$

Since $\lambda\in C^2\left(\bar{\Omega}\right)$, $e^{-\lambda}$ is bounded, which means that $e^{-\lambda}\le C$. Combined with $u\in H_0^1(\Omega)$, we know that $w(x)=e^{-\lambda}u\in L^2(\Omega)$. Similarly, since $\mathbf{V}\in C^1\left(\bar{\Omega}\right)^N$, it is also bounded, we can also derive $\nabla w\in L^2(\Omega)$. Therefore, $w(x)\in H^1(\Omega)$. Recall the boundary condition $w(x)=0$, it can be concluded that $w(x)\in H_0^1(\Omega)$. Therefore, we can derive the variational formulation associated with 3.1: Find $w\in H_0^1(\Omega)$ s.t. for any $v\in H_0^1(\Omega)$,

$$a(w,v)=L(v) \tag{45}$$

where

$$a(w,v)=\nu\int_\Omega\nabla w\cdot\nabla v+\int_\Omega\left(\frac{1}{4\nu}|\mathbf{V}|^2-\frac{1}{2}\operatorname{div}(\mathbf{V})+\alpha\right)w\cdot v\,d\mathbf{x} \tag{46}$$

and

$$L(v)=\int_\Omega e^{-\lambda}fv\,d\mathbf{x}. \tag{47}$$

Theorem 3.10. Let $C_1>0$ be the constant of the Poincare's inequality (21) and $B:=\max_{\mathbf{x}\in\bar{\Omega}}|V(\mathbf{x})|^2$. Under the condition that either $\alpha>0$, or $\alpha=0$ and $\nu>\frac{\sqrt{C_1B}}{2}$, a unique solution exists for the advection–diffusion equation (10) with Dirichlet boundary conditions.

Proof. Still, we can easily see the linearity of $L(\cdot)$ and $a(\cdot,\cdot)$. First, we prove the continuity of $L(\cdot)$ and $a(\cdot,\cdot)$. We begin with analyzing $L(\cdot)$:

DOI: 10.54254/2753-8818/2026.CH31164

$$\begin{aligned} |L(v)| = \left|\int_\Omega e^{-\lambda} fv\, d\mathbf{x}\right| &\leq \int_\Omega \left|e^{-\lambda} fv\right| d\mathbf{x} \\ &\leq C \int_\Omega |fv|\, d\mathbf{x} \\ &\leq C\|f\|_{L^2(\Omega)} \| v \|_{L^2(\Omega)} \\ &\leq CC_1C_2 \| v \|_{H_0^1(\Omega)} \end{aligned} \tag{48}$$

In which $C > 0$ is the constant that controls the magnitude of $e^{-\lambda}$, $C_1 = \|f\|_{L^2(\Omega)}$ and C_2 denotes the constant in Poincare's Inequality. Since $CC_1C_2 > 0$, $L(v)$ is continuous. We also have

$$\begin{aligned} |a(w,v)| &= \left|\nu \int_\Omega \nabla w \cdot \nabla v + \int_\Omega \left(\tfrac{1}{4\nu}|\mathbf{V}|^2 - \tfrac{1}{2}\operatorname{div}(\mathbf{V}) + \alpha\right) w \cdot v\, d\mathbf{x}\right| \\ &\leq \nu \left|\int_\Omega \nabla w \cdot \nabla v\right| + \left|\int_\Omega \left(\tfrac{1}{4\nu}|\mathbf{V}|^2 - \tfrac{1}{2}\operatorname{div}(\mathbf{V}) + \alpha\right) w \cdot v\, d\mathbf{x}\right|. \end{aligned} \tag{49}$$

We analyze the two terms. For the first term, we still have

$$\nu \left|\int_\Omega \nabla u \cdot \nabla v d\mathbf{x}\right| \leq \nu\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} = \nu \| u \|_{H_0^1(\Omega)} \| v \|_{H_0^1(\Omega)}. \tag{50}$$

For the second term, since $\mathbf{V} \in C^1\left(\bar{\Omega}\right)$, both $\mathbf{V}$ and $\operatorname{div}\mathbf{V}$ are bounded. Therefore, we introduce a constant $C_3 > 0$ that controls the magnitude of $\frac{1}{4\nu}|\mathbf{V}|^2 - \frac{1}{2}\operatorname{div}(\mathbf{V}) + \alpha$, satisfying

$$\tfrac{1}{4\nu}|\mathbf{V}|^2 - \tfrac{1}{2}\operatorname{div}(\mathbf{V}) + \alpha \leq C_3. \tag{51}$$

Now we have

$$\begin{aligned} \left|\int_\Omega \left(\tfrac{1}{4\nu}|\mathbf{V}|^2 - \tfrac{1}{2}\operatorname{div}(\mathbf{V}) + \alpha\right) w \cdot v\, d\mathbf{x}\right| &\leq C_3 \left|\int_\Omega wv\, d\mathbf{x}\right| \\ &\leq C_3 \| w \|_{L^2(\Omega)} \| v \|_{L^2(\Omega)} \\ &\leq C_3C' \| w \|_{H_0^1(\Omega)} \| v \|_{H_0^1(\Omega)} \end{aligned} \tag{52}$$

where the above inequalities are satisfied by Cauchy-Schwarz Inequality and Poincare's Inequality, respectively. Here C' is the product of the two constants for w and v in Poincare's Inequality. Therefore,

$$a(w,v) \leq (CC_1C_2 + C_3C') \| w \|_{H_0^1(\Omega)} \| v \|_{H_0^1(\Omega)}. \tag{53}$$

Since $CC_1C_2 + C_3C' > 0$, $a(\cdot,\cdot)$ must also be continuous, which completes our proof.

Next, we will show the coercivity of $a(\cdot,\cdot)$ in the variational formulation:

$$a(v,v) = \nu \int_\Omega |\nabla v|^2\, d\mathbf{x} + \int_\Omega \left(\tfrac{1}{4\nu}|\mathbf{V}|^2 - \tfrac{1}{2}\operatorname{div}(\mathbf{V}) + \alpha\right) v^2\, d\mathbf{x}. \tag{54}$$

We will deal with the negative term $-\frac{1}{2}\int_\Omega \operatorname{div}(\mathbf{V}) v^2\, d\mathbf{x}$ first. Applying Lemma 3.5 to the term gives

$$-\frac{1}{2}\int_{\Omega}\operatorname{div}(\mathbf{V})v^2\,d\mathbf{x} = \int_{\Omega}\nabla v\cdot(\mathbf{V}v)\,d\mathbf{x}. \tag{55}$$

Notice that

$$\begin{aligned}\left|\int_{\Omega}\nabla v\cdot(\mathbf{V}v)\,d\mathbf{x}\right| &\leq \|\nabla v\|_{L^2}\|\mathbf{V}v\|_{L^2} \\ &\leq \frac{c}{2}\int|\nabla v|^2\,d\mathbf{x} + \frac{1}{2c}\int_{\Omega}|\mathbf{V}|^2v^2\,d\mathbf{x},\end{aligned} \tag{56}$$

where the third inequality is satisfied by Young's Inequality. Combining it into $a(v,v)$ gives

$$a(v,v) \leq \left(\nu - \frac{c}{2}\right)\int_{\Omega}|\nabla v|^2 d\mathbf{x} + \left(\frac{1}{4\nu} - \frac{1}{2c}\right)\int_{\Omega}|\mathbf{V}|^2v^2 d\mathbf{x} + \alpha\int_{\Omega}v^2 d\mathbf{x}. \tag{57}$$

It needs the first term to be positive to guarantee the coercivity. Then $\nu - \frac{c}{2} > 0$ implies that $c < 2\nu$. For the next two terms, we need to consider different cases: $\alpha > 0$ or $\alpha = 0$. When $\alpha < 0$, the coercivity cannot be guaranteed. First, we consider the case when $\alpha > 0$. Now the second term has to be non-negative to satisfy coercivity:

$$\frac{|\mathbf{V}|^2}{4\nu} - \frac{|\mathbf{V}|^2}{2c} + \alpha \geq 0 \Leftrightarrow c \geq \frac{|\mathbf{V}|^2}{\frac{|\mathbf{V}|^2}{2\nu} + 2\alpha}. \tag{58}$$

Take $\gamma = \max_{x\in\bar{\Omega}}\frac{|\mathbf{V}|^2}{\frac{|\mathbf{V}|^2}{2\nu}+2\alpha}$, then $c \geq \gamma$. Since $\mathbf{V}$ is bounded and $\alpha > 0$, we can notice that $\gamma < 2\nu$. Recall that $c < 2\nu$, there must exist a constant c such that $\gamma \leq c < 2\nu$, and taking $c = \gamma$ satisfy the condition. We're guranteed to have

$$\begin{aligned} a(v,v) &\geq \left(\nu - \frac{\gamma}{2}\right)\int_{\Omega}|\nabla v|^2 d\mathbf{x} \\ &\geq \left(\nu - \frac{\gamma}{2}\right)\|v\|^2_{H_0^1(\Omega)}.\end{aligned} \tag{59}$$

So in this case $(\alpha > 0)$ we have proved that $a(\cdot,\cdot)$ is coercive. This ensures uniqueness of a solution to (23) when $\alpha > 0$.

In the other case when $\alpha = 0$, we still have

$$a(v,v) \leq \left(\nu - \frac{c}{2}\right)\int_{\Omega}|\nabla v|^2 d\mathbf{x} + \left(\frac{1}{4\nu} - \frac{1}{2c}\right)\int_{\Omega}|\mathbf{V}|^2v^2 d\mathbf{x}. \tag{60}$$

The Poincare's inequality can be used to bound the second term

$$\int_{\Omega}v^2 d\mathbf{x} \leq C_1\int|\nabla v|^2 d\mathbf{x}, \tag{61}$$

Since $\mathbf{V}$ is bounded, we introduce a constant $B := \max_{\mathbf{x}\in\bar{\Omega}}|V(\mathbf{x})|^2$. Now we have

$$\left(\frac{1}{4\nu} - \frac{1}{2c}\right)\int_{\Omega}|\mathbf{V}|^2v^2 d\mathbf{x} < C_1B\left(\frac{1}{4\nu} - \frac{1}{2\nu}\right)\int|\nabla v|^2 d\mathbf{x}. \tag{62}$$

Since this term could be negative, we need the following inequality hold to guarantee coercivity:

$$\nu - \frac{c}{2} > C_1B\left(\frac{1}{4\nu} - \frac{1}{2\nu}\right). \tag{63}$$

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Rearranging the terms gives

$$\nu + \tfrac{C_1 B}{4\nu} > \tfrac{c}{2} + \tfrac{C_1 B}{2c}. \tag{64}$$

Let $t = C_1 B$. Let constant $K = \nu + \frac{t}{4\nu}$, $f(c) = \frac{c}{2} + \frac{t}{2c}$. By examining the first-order derivative and second-order derivative, we can verify that $f(c)$ reached its minimum value when $c = \sqrt{t}$. By AM-GM inequality we have

$$K = \nu + \tfrac{t}{4\nu} \geq 2\sqrt{\nu \cdot \tfrac{t}{4\nu}} = \sqrt{t}, \tag{65}$$

with the equality holds if and only if $\nu = \frac{t}{4\nu}$.

The inequality (3.9) requires $K > \sqrt{t}$ to hold, since $f(c) \geq \sqrt{t}$. That is,

$$\nu + \tfrac{t}{4\nu} > \sqrt{t}. \tag{66}$$

When $t < 4\nu^2$, the inequality strictly holds. Therefore, for any $c \in \left(\frac{t}{2\nu}, 2\nu\right)$, $K > f(c)$. Since $t < 4\nu^2$, which implies that $\frac{t}{2\nu} < 2\nu$, so $\left(\frac{t}{2\nu}, 2\nu\right)$ is nonempty, there must exist such c that satisfies the condition. We can easily verify that the inequality doesn't strictly hold when $t \geq 4\nu^2$

When $t = 4\nu^2$, then $K = 2\nu$, $f(c) \geq 2\nu$, with equality holds iff $c = 2\nu$. The inequality doesn't strictly hold.

When $t > 4\nu^2$, we need $c \in \left(2\nu, \frac{t}{2\nu}\right)$ for the inequality to hold. But since $c < 2\nu$, the inequality doesn't hold.

Therefore, if $\alpha = 0$ and $\nu > \frac{\sqrt{C_1 B}}{2}$, the coercivity is also satisfied, which ensures an unique solution to the original equation (13). Since $w = e^{\lambda} u$, where u is a solution of equation (10), therefore the equation (10) also has a unique solution.

3.5. Dependence of the solution on $(V\alpha)$ and its continuity

We investigate the continuous dependence of the unique solution on the parameters $(V\alpha)$. Throughout this section, the diffusion coefficient $\nu > 0$ is fixed, and the velocity field $V \in C^1\left(\bar{\Omega}\right)^n$ is assumed to be divergence free, i.e., $\mathrm{div} V = 0$.

For given parameters $(\nu V \alpha)$, w e consider the mapping $(V, \alpha) \longmapsto u_{\nu,V,\alpha}$, from $C^1\left(\bar{\Omega}\right)^n \times \mathbb{R}_{\geq 0}$ into $H_0^1(\Omega)$, and show that it is continuous. Specifically, for any divergence-free $V_0 \in C^1\left(\bar{\Omega}\right)^n$ and $\alpha_0 > 0$, then

$$\|u_{\nu,\mathbf{V},\alpha} - u_{\nu,\mathbf{V}_0,\alpha_0}\|_{H_0^1(\Omega)} \leq C\left(\max_{\mathbf{x}\in\bar{\Omega}} |\mathbf{V}(\mathbf{x}) - \mathbf{V}_0(\mathbf{x})| + |\alpha - \alpha_0|\right). \tag{67}$$

We first provide a corollary, and then we give the proof.

Corollary 3.11. Let $V, V_0 \in C^1\left(\bar{\Omega}\right)^n$ and $u_0 \in H_0^1(\Omega)$.

$$\begin{aligned}&\parallel (\mathbf{V}_0-\mathbf{V})\cdot\nabla u_0 \parallel_{L^2(\Omega)} \quad \parallel u-u_0 \parallel_{L^2(\Omega)} + |\alpha_0-\alpha| \ \parallel u_0 \parallel_{L^2(\Omega)} \quad \parallel u-u_0 \parallel_{L^2(\Omega)} \\ &\qquad \le C\left(\max_{x\in\bar{\Omega}}|(\mathbf{V}-\mathbf{V}_0)(\mathbf{x})|+|\alpha-\alpha_0|\right) \parallel u-u_0 \parallel_{L^2(\Omega)}.\end{aligned} \tag{68}$$

Proof. Since $\nabla u_0 \in L^2(\Omega)$, then

$$\begin{aligned}\parallel (\mathbf{V}_0-\mathbf{V})\cdot\nabla u_0\|_{L^2(\Omega)} &\le A\|\mathbf{V}_0-\mathbf{V}\|_{L^2(\Omega)} \\ &= A\sqrt{\int_\Omega |\mathbf{V}_0-\mathbf{V}|^2 d\mathbf{x}} \\ &\le A\sqrt{\int_\Omega \left(\max_{x\in\bar{\Omega}}|(\mathbf{V}-\mathbf{V}_0)(\mathbf{x})|\right)^2 d\mathbf{x}} \\ &\le \frac{A}{\sqrt{\lambda(\Omega)}}\max_{x\in\bar{\Omega}}|(\mathbf{V}-\mathbf{V}_0)(\mathbf{x})|,\end{aligned} \tag{69}$$

The desired estimate follows directly by combining this bound with Hölder's inequality.

Theorem 3.12. Let $\operatorname{div}\mathbf{V}=0$. Fix a diffusion coefficient $\nu>0$. The mapping

$$(\mathbf{V},\alpha)\mapsto u_{\nu,\mathbf{V},\alpha}, \tag{70}$$

which acts from $C^1\left(\bar{\Omega}\right)^n \times \mathbb{R}_{\ge 0}$ into $H_0^1(\Omega)$, is continuous.

Proof. Let $u=u_{\nu,V,\alpha}$, $u_0=u_{\nu,V_0,\alpha_0}$, and denote by $a(\cdot,\cdot)$ and $a_0(\cdot,\cdot)$ the corresponding bilinear forms. By definition,

$$a(u,v)=L(v),\, a_0(u_0,v)=L(v) \tag{71}$$

Subtracting these identities yields,

$$a(u-u_0,v)=(a_0-a)(u_0,v) \tag{72}$$

for any $v\in H_0^1(\Omega)$. Choosing $v=u-u_0$, we obtain

$$C_\nu\|u-u_0\|_{H_0^1(\Omega)}^2 \le |(a_0-a)(u_0,u-u_0)| \tag{73}$$

Observing that

$$(a_0-a)(u_0,v)=\int_\Omega (\mathbf{V}_0-\mathbf{V})\cdot\nabla u_0 v d\mathbf{x}+(\alpha_0-\alpha)\int_\Omega u_0 v d\mathbf{x} \tag{74}$$

Hölder's inequality and Corollary 3.11 imply

$$|(a_0-a)(u_0,u-u_0)| \le C\left(\max_{x\in\bar{\Omega}}|(\mathbf{V}-\mathbf{V}_0)(\mathbf{x})|+|\alpha-\alpha_0|\right)\|u-u_0\|_{L^2(\Omega)} \tag{75}$$

Applying Poincaré's inequality and combining the above estimates yields

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$$\|u-u_0\|_{H_0^1(\Omega)} \leq \frac{CC_1}{C_\nu}\left(\max_{x\in\bar{\Omega}}|(\mathbf{V}-\mathbf{V}_0)(\mathbf{x})|+|\alpha-\alpha_0|\right), \tag{76}$$

which completes the proof.

3.6. Existence of the zero-diffusion limit

In this subsection, we simplify the problem, considering the problem in one dimension. Then, $\mathbf{V}$ and $\mathbf{x}$ are no longer vectors, but numbers, so we rewrite them as V and x, respectively. For simplicity's sake, let $\Omega=(0,1)$ and fix V and $\alpha=0$. Recall the condition $\mathrm{div}V=0$: now that $V:\mathbb{R}\to\mathbb{R}$, $\mathrm{div}V=0 \Leftrightarrow V(x)\equiv C$ for some constant C for any $x\in(0,1)$. The advection–diffusion equation (10) now collapses to the ordinary differential equation

$$\nu u'' + Vu' = f, \quad x\in(0,1). \tag{77}$$

Consider the mapping $\nu\mapsto u_\nu=u_{\nu,V,0}$. Define the zero-diffusion limit as following.

Definition 3.13. If u_ν admits a limit in $H_0^1(\Omega)$ as $\nu\to 0^+$, i.e. we have $u\in H_0^1(\Omega)$ which satisfies

$$\|u_\nu-u\|_{H_0^1(\Omega)}\to 0, \quad \nu\to 0^+, \tag{78}$$

then we call u the zero-diffusion limit of u_ν.

Recall that u_ν is the solution to (77), and it satisfies variational formulation:

$$\nu\int_0^1 u_\nu' v' dx+\int_0^1 Vu_\nu' v dx=\int_0^1 fv dx, \quad \text{for any} v\in H_0^1(0,1). \tag{79}$$

Assuming that the zero-diffusion limit exists, taking the limit $\nu\to 0^+$ on both sides yields

$$Vu' = f. \tag{80}$$

Therefore,

$$\int_0^1 f(x)dx=\int_0^1 Vu' dx = Vu(x)\Big|_0^1 = V(u(1)-u(0))=0. \tag{81}$$

The satisfaction of this equation is necessary (but not sufficient) for the zero-diffusion limit to exist. The following theorem presents the full set of conditions required and sufficient to guarantee this limiting behavior.

Theorem 3.14. Assume f is a non-zero function with bounded $f'(x)$ on (0,1). A necessary and sufficient condition for the zero-diffusion limit to exist is that

$$\int_0^1 f(x)dx=0 \quad \text{and} \quad V\neq 0. \tag{82}$$

Proof. We have proved that

$$\int_0^1 f(x)dx=0 \tag{83}$$

is a necessary condition for the existence of the zero-diffusion limit at the beginning of this section.

Also, our previous result suggests that u_ν satisfies the equation

$$\nu u_\nu'' + V u_\nu' = f, \tag{84}$$

and u satisfies the equation

$$V u' = f. \tag{85}$$

Subtracting the above two equations and letting $w_\nu = u_\nu - u$, we have

$$-\nu w_\nu'' + V w_\nu' = \nu u'' \Leftrightarrow w_\nu'' - \frac{V}{\nu} w_\nu' = -u''. \tag{86}$$

In this case, we assume that $V \neq 0$, and integration on (80) gives the expression of u:

$$u = \frac{1}{V} \int_0^x f(y) dy. \tag{87}$$

Substituting $x = 0$ to the expression of u gives a boundary condition that $u(0) = 0$. Combining with our necessary condition that $\int_0^1 f(x) dx = 0$, we have the other condition that $u(1) = 0$. Recall the boundary condition of u_ν on $[0, 1]$:

$$u_\nu(0) = u_\nu(1) = 0. \tag{88}$$

Together we will get the boundary condition of w_ν, since $w_\nu = u_\nu - u$:

$$w_\nu(0) = w_\nu(1) = 0. \tag{89}$$

Since $u' = \frac{f}{V}$, differentiation again gives the second derivative of u

$$u'' = \frac{f'}{V}. \tag{90}$$

Substituting into (80) gives

$$-\nu w_\nu'' + V w_\nu' = \nu \frac{f'}{V}. \tag{91}$$

Letting $g_\nu = w_\nu'$ gives

$$g_\nu' - \frac{V}{\nu} g_\nu = -\frac{f'}{V}. \tag{92}$$

We let $\lambda = \frac{V}{\nu}$ for clarity, and multiply $e^{-\lambda x}$ on both sides:

$$e^{-\lambda x}(g_\nu' - \lambda g_\nu) = -e^{-\lambda x}\frac{f'}{V} \Leftrightarrow \frac{d}{dx}\left(e^{-\lambda x} g_\nu\right) = -\frac{f'}{V} e^{-\lambda x}. \tag{93}$$

Then, we integrate both sides from 0 to 1:

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$$\int_0^x \frac{d}{dx}\left(e^{-\lambda x} g_\nu\right) = -\int_0^x \frac{f'}{V} e^{-\lambda s} ds. \tag{94}$$

Solving the integral on the left gives the expression of $g_\nu(x)$, which is also equivalent to $w_\nu'(x)$:

$$w_\nu'(x) = g_\nu(x) = Ce^{\lambda x} - \frac{e^{\lambda x}}{V} \int_0^x f'(s) e^{-\lambda s} ds,\ C \in \mathbb{R}. \tag{95}$$

We define $w_\nu(x)$ as

$$\begin{aligned} w_\nu(x) &= \int_0^x w_\nu'(s) ds \\ &= \int_0^x Ce^{\lambda s} ds - \int_0^x \frac{e^{\lambda s}}{V} \int_0^s f'(t) e^{-\lambda t} dt\, ds. \end{aligned} \tag{96}$$

It's clear that $\int_0^x Ce^{\lambda s} ds = C\frac{e^{\lambda x}-1}{\lambda}$. To simplify the double integral, we interchange the order of integration:

$$\begin{aligned} \int_0^x \frac{e^{\lambda s}}{V} \int_0^s f'(t) e^{-\lambda t} dt\, ds &= \int_0^x f'(t) e^{-\lambda t} \int_t^x \frac{e^{\lambda s}}{V} ds\, dt \\ &= \int_0^x f'(t) e^{-\lambda t} \frac{e^{\lambda x} - e^{\lambda t}}{\lambda V} dt \\ &= \frac{1}{\lambda V} \int_0^x f'(t)\left(e^{\lambda(x-t)} - 1\right) dt. \end{aligned} \tag{97}$$

Substituting back into $w_\nu(x)$ gives

$$w_\nu(x) = C\frac{e^{\lambda x}-1}{\lambda} - \frac{1}{\lambda V} \int_0^x f'(t)\left(e^{\lambda(x-t)} - 1\right) dt. \tag{98}$$

Now we check the boundary conditions. Note that this expression automatically satisfies the boundary condition when $x = 0$ since $w_\nu(0) = 0$. When $x = 1$, we must have $w_\nu(1) = 0$, which is

$$w_\nu(1) = C\frac{e^{\lambda}-1}{\lambda} - \frac{1}{\lambda V} \int_0^1 f'(t)\left(e^{\lambda(1-t)} - 1\right) dt = 0. \tag{99}$$

We can obtain an expression of C from the equation:

$$C = \frac{1}{V(e^\lambda - 1)} \int_0^1 f'(t)\left(e^{\lambda(1-t)} - 1\right) dt. \tag{100}$$

Also, suppose that $f'(t)$ is bounded. Then, there is some $C_1 \in \mathbb{R}$ satisfying $| f'(t) | \leq C_1$.

Next, we examine whether the zero-diffusion limit of u_ν exists, meaning $\| w_\nu' \|_{L^2(\Omega)} = \| u_\nu - u \|_{H_0^1(\Omega)} \to 0$ as $\nu \to 0$, for the three scenarios: V is positive, V is negative and V is zero.

Case 1 - $V > 0$ Combining the previous representations yields the following equivalent expression for $w_\nu'(x)$:

$$w_\nu'(x) = \frac{e^{\lambda x}}{V} \int_x^1 f'(t) e^{-\lambda t} dt + \frac{e^{\lambda x}}{V(e^\lambda - 1)} \int_0^1 f'(t) e^{-\lambda t} dt - \frac{e^{\lambda x}}{V(e^\lambda - 1)} \int_0^1 f'(t) dt. \tag{101}$$

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To simplify notation, the following three represent the three terms in the preceding formula. That is,

$$F_1(x) = \frac{e^{\lambda x}}{V} \int_x^1 f'(t) e^{-\lambda t} dt \tag{102}$$

$$F_2(x) = \frac{e^{\lambda x}}{V(e^{\lambda}-1)} \int_0^1 f'(t) e^{-\lambda t} dt \tag{103}$$

$$F_3(x) = \frac{e^{\lambda x}}{V(e^{\lambda}-1)} (f(1) - f(0)). \tag{104}$$

Next, we will separately prove that as $\nu \to 0$, $\lambda \to \infty$, the L^2 norms above all tend to zero. Together we will prove that $w_\nu'(x) \to 0$.

Regarding $F_1(x)$, the fact that $f'(t)$ is bounded implies that

$$\begin{aligned} \left| \int_x^1 f'(t) e^{-\lambda t} dt \right| &\le C_1 \int_x^1 e^{-\lambda t} dt \\ &\le \frac{C_1}{\lambda} e^{-\lambda x}. \end{aligned} \tag{105}$$

Therefore

$$e^{\lambda x} \left| \int_x^1 f'(t) e^{-\lambda t} dt \right| \le \frac{C_1}{\lambda} . \tag{106}$$

Now, we can get:

$$\begin{aligned} \|F_1(x)\|_{L^2}^2 &= \int_0^1 (F_1(x))^2 dx \\ &= \int_0^1 \frac{e^{2\lambda x}}{V^2} \left(\int_0^1 f'(t) e^{-\lambda t} dt \right)^2 dx \\ &\le \int_0^1 \frac{(C_1)^2}{\lambda^2 V^2} dx = \frac{(C_1)^2}{\lambda^2 V^2} . \end{aligned} \tag{107}$$

Since C_1 and V are both bounded constants, we have

$$\lim_{\lambda \to \infty} \frac{(C_1)^2}{\lambda^2 V^2} = 0. \tag{108}$$

Then it follows that

$$\lim_{\lambda \to \infty} \|F_1(x)\|_{L^2}^2 = 0 \tag{109}$$

and

$$\lim_{\lambda \to \infty} \|F_1(x)\|_{L^2} = 0. \tag{110}$$

For $F_2(x)$, we have

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$$\begin{aligned} \|F_1(x)\|_{L^2}^2 &= \frac{\left(\int_0^1 f'(t)e^{-\lambda t}dt\right)^2}{V^2(e^\lambda-1)^2} \int_0^1 e^{2\lambda x}dx \\ &= \frac{\left(\int_0^1 f'(t)e^{-\lambda t}dt\right)^2}{V^2} \cdot \frac{e^{2\lambda}-1}{(e^\lambda-1)^2}. \end{aligned} \tag{111}$$

when $f'(t)$ is bounded, then we can get

$$\lim_{\lambda\to\infty} \frac{\left(\int_0^1 f'(t)e^{-\lambda t}dt\right)^2}{V^2} = 0. \tag{112}$$

Therefore,

$$\lim_{\lambda\to\infty} \|F_2(x)\|_{L^2} = 0. \tag{113}$$

For $F_3(x)$, we can get

$$\begin{aligned} \|F_3(x)\|_{L^2}^2 &= \frac{(f(1)-f(0))^2}{V^2(e^\lambda-1)^2} \int_0^1 e^{2\lambda x}dx \\ &= \frac{(f(1)-f(0))^2}{2\lambda V} \cdot \frac{e^{2\lambda}-1}{(e^\lambda-1)^2}. \end{aligned} \tag{114}$$

We have shown that $\lim_{\lambda\to\infty}\frac{e^{2\lambda}-1}{(e^\lambda-1)^2} = 1$ in the previous case. And since $f(1), f(0)$ and V are all bounded constants, $\lim_{\lambda\to\infty}\frac{(f(1)-f(0))^2}{2\lambda V} = 0$. Therefore, we have

$$\lim_{\lambda\to\infty} \|F_3(x)\|_{L^2} = 0. \tag{115}$$

Finally, since the L^2 norms all tend to 0 as $\nu\to 0$, $\|w_\nu'\|_{L^2}$ also converges at 0, and the zero-diffusion limit exists.

Case 2 - $V<0$ In this case, as $\nu\to 0$, $\lambda\to-\infty$. Recall that from (101), we can express $w_\nu'(x)$ as the sum of $F_4(x)$ and $F_5(x)$, where

$$\begin{aligned} F_4(x) &= Ce^{\lambda x} \\ F_5(x) &= -\frac{e^{\lambda x}}{V}\int_0^x f'(s)e^{-\lambda s}ds. \end{aligned} \tag{116}$$

What's more, we will show that $\|F_4(x)\|_{L^2}$ and $\|F_5(x)\|_{L^2}$ all tend to 0 as $\lambda\to-\infty$, then $\|w_\nu'(x)\|_{L^2}\to 0$.

We know that

$$\|F_4(x)\|_{L_2}^2 = \int_0^1 C^2e^{2\lambda x}dx = \frac{C^2}{2\lambda}\left(e^\lambda-1\right). \tag{117}$$

C becomes a bounded constant, since when $\lambda\to-\infty$, $e^{\lambda(1-t)}\to 0$. Also, $e^\lambda\to 0$ as $\lambda\to-\infty$. Therefore, we still have $\lim_{\lambda\to-\infty}\|F_4(x)\|_{L_2}^2 = 0$ and $\lim_{\lambda\to-\infty}\|F_4(x)\|_{L_2} = 0$.

Then we consider $F_5(x)$. We have

$$|F_5\,(x)| \leq \frac{e^{\lambda x}}{|V|} \int_0^x |f'\,(s)|e^{-\lambda s}ds \leq \frac{C_1 e^{\lambda x}}{|V|\lambda} \left(1 - e^{-\lambda x}\right). \tag{118}$$

Consequently,

$$\|F_5\,(x)\|_{L_2}^2 \leq \frac{(C_1)^2}{V^2\lambda^2}\,. \tag{119}$$

Hence,

$$\lim_{\lambda\to-\infty} \|F_5\,(x)\|_{L_2} = 0. \tag{120}$$

Finally, we have $\|F_4\,(x)\|_{L_2}$ and $\|F_5\,(x)\|_{L_2}$ all tend to 0 as $\lambda \to -\infty$, therefore $\|w_\nu'\,(x)\|_{L^2} \to 0$, which means that the zero-diffusion limit exists.

Case 3 - $V = 0$

In this case, the equation (77) becomes

$$-\nu u_\nu'' = f \;\; (f \neq 0). \tag{121}$$

Integration on u_ν'' gives

$$u_\nu' = -\frac{1}{\nu} \int_0^x f\,(s)ds + C_2 \tag{122}$$

and

$$u_\nu = -\frac{1}{\nu} \int_0^x \int_0^t f\,(s)ds\;dt + C_2 x + C_3. \tag{123}$$

We must have

$$C_3 = 0,\; C_2 = \frac{1}{\nu} \int_0^1 \int_0^t f\,(s)ds\;dt. \tag{124}$$

Substituting C_2 back into u_ν' gives

$$u_\nu' = -\frac{1}{\nu} \int_0^x f\,(s)ds + \frac{1}{\nu} \int_0^1 \int_0^t f\,(s)ds\;dt. \tag{125}$$

We can see that u_ν diverges as $\nu \to 0$. So, when $V = 0$, the zero-diffusion limit doesn't exist.

Now we have completed our proof.

4. Conclusion

We analyzed the steady advection–diffusion equation in a variational framework and identified several explicit conditions guaranteeing coercivity and hence well-posedness. For divergence-free flows, coercivity holds unconditionally; for general vector fields, a sharp bound on div V ensures stability. In the gradient-field case, an exponential change of variables transforms the problem into a symmetric elliptic equation, yielding well-posedness under natural assumptions. We also established continuity of solutions with respect to the velocity field and reaction parameter, and derived a necessary and sufficient condition for the zero-diffusion limit in one dimension. These results

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provide a unified view of steady transport–diffusion models and suggest extensions to non-homogeneous boundaries, lower-regularity flows, and time-dependent settings.

Author contribution

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