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The Motion of Pendulum with and Without Air Resistance and the Spring

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Abstract. This paper mainly discusses the simple pendulum without air resistance, the simple pendulum with air resistance, and the spring pendulum without air resistance. For a simple pendulum without air resistance, we give the analytical solutions (small angle approximation, $\sin\theta = \theta$), and we give numerical solutions for pendulums in all cases. Forward Difference method and RK4 are used in the part of numerical solutions and sensitivity tests. We also compare the difference between analytical solutions and numerical solutions for a simple pendulum without air resistance, and the difference between a simple pendulum with and without air resistance, and analyze how these differences are engendered. This research uses Python to visually reveal the motion of pendulums and the difference between them. An ideal pendulum shows a periodic motion; the amplitude of a simple pendulum with air resistance decreases to zero with a slight period increase; the motion of a spring pendulum is irregular. Sensitivity tests indicate how the length of the rod, the gravitational parameter, the initial angle, and the spring parameter (for the spring pendulum) influence the motion of the pendulum.

Keywords: simple pendulum, spring pendulum, air resistance, numerical model, Python.

1. Introduction

Over several thousand years, physics has developed swiftly, and the pendulum is a sub-branch of physics that has been researched and analyzed for several years. The pendulum has been divided into several types, such as the simple pendulum, the elastic pendulum, and the flexible pendulum [1]. This paper wants to explore the simple pendulum with and without air resistance and the spring.

A simple pendulum is a classic physics system consisting of a mass suspended from a fixed point, swinging under gravity, which includes basic forces: the restoring force, driving the oscillation; the tension component, balancing part of gravitational pull [2]. For the simple pendulum with air resistance, there is an additional drag force, which opposes motion. For the spring pendulum model, the rig rod is replaced by an elastic spring, introducing vertical oscillation alongside swinging motion.

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In an idealized scenario, research always studies a pendulum without air resistance, simplifying the analysis. However, in reality, air resistance plays a significant role, causing energy dissipation and altering the pendulum's motion over time. Unlike air resistance, which simply slows the pendulum down, the spring transforms how energy moves through the system. The bouncing motion stores and releases energy differently than pure swinging. Thus, this research paper wants to figure out the effects of air resistance and the combination of bouncing motion and swinging motion.

To achieve this, this research uses a comprehensive numerical model that integrates the equations of motion for all three cases: the ideal pendulum, the damped pendulum, and the spring pendulum. While analytical solutions suffice for the ideal case, the nonlinearities and coupled dynamics present in the damped and spring systems make numerical integration the only practical approach for a comprehensive analysis. This necessity to resort to numerical models for solving nonlinear differential equations is well-established in the literature of dynamical systems [3]. This framework allows us to simulate and compare the long-term behavior, and trajectory of each system under identical initial conditions, providing insights that would be impossible to obtain through purely analytical means. The equations will be solved by algorithms. The reliability of the numerical results will be further strengthened through a sensitivity analysis, which quantifies the impact of variations in input parameters like initial angle and damping coefficient on the model's output.

2. Literature review

This work builds on foundational studies. The dynamics of oscillatory systems have been studied, particularly in the context of simplifying assumptions and their limitations. One key approximation in the analysis of pendulum motion is the small angle approximation, which assumes $\sin\theta \approx \theta$ for negligible amplitudes. Nelson and Olsson [2] explored this approximation through perturbation theory, demonstrating that while it holds this assumption rigorously, large amplitudes make the oscillation period depend on the angle size. Their work highlights how higher-order corrections become significant and necessary as the amplitude grows, requiring more sophisticated analytical and numerical approaches.

For systems where analytical solutions are intractable, numerical methods provide a practical alternative. LeVeque [4] elaborated finite difference methods as a robust framework for discretizing differential equations, enabling the simulation of nonlinear oscillators with initial conditions. These methods are useful for the situations that are complicated to be solved by analytical solutions, including the situation with air resistance and larger amplitudes.

This research paper includes dissipative effects, such as linear damping due to air resistance, further complicating the analysis. Garcia [5] employed Euler method to model damping effects, illustrating how even first-order numerical techniques can capture qualitative trends in energy dissipations with limitations in accuracy and stability. This underscores trade-offs between simplicity and fidelity in modeling real-world oscillatory systems.

Collectively, these studies delineate a progression from idealized models to realistic scenarios, emphasizing the role of perturbation theory and numerical methods in bridging the gap between the theory and algorithms.

3. Basic theory and model construction

An important part in the field of mechanical research, the pendulum is a system of an object suspended by an inextensible light string and it can be considered as simple harmonic motion as the angle is small enough. A spring pendulum, on the other hand, is a system of an object suspended by

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a spring. On the foundation of the simple pendulum, the spring pendulum will also be affected by the change because of the extension and contraction of the spring and thus has a more complex motion.

For the pendulum without air resistance, the motion equation, which can be derived from Newton's Second law, is

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L} sin\theta. \tag{1}$$

In this equation, θ is the angle, t is the time, L is the length of the string, and g is the gravitational coefficient. When the angle is small enough, we replace sinθ as θ and get the equation:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta. \tag{2}$$

As for the analytical solution, which can be derived from the conclusion of simple harmonic motion, is

$$\theta(t) = \theta_0 cos(\omega t + \varphi) \tag{3}$$

where ω = $\sqrt{\frac{g}{l}}$ is the angular frequency and φ is the phase constant.

For a pendulum without air resistance, let β be the resistance coefficient, the equation will be:

$$-mgsin\theta - \beta\omega = mL. \tag{4}$$

In the case of the spring pendulum, let l0 be the initial length, k be the spring coefficient, r be the current length of the spring and m be the mass of the suspended object, the equations will be:

$$\frac{\partial^2 r}{\partial t^2} - r * \frac{\partial\theta}{\partial t} + \frac{k}{m}(r - l_0) - g\cos\theta = 0 \tag{5}$$

$$\frac{\partial^2\theta}{\partial t^2} + \frac{2}{r}\frac{\partial r}{\partial t}\frac{\partial\theta}{\partial t} + \frac{s}{r}\sin\theta = 0 \tag{6}$$

The motion equations of spring pendulum can be derived with Lagrangian and Lagrange equation.

4. Numerical model

4.1. Forward difference method

The forward difference method, also known as Euler method, is a simple method for discrete equations by approximating derivative. Take a simple example: for a first-order derivative $\frac{dy}{dt}$, with

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a time step of Δt, the approximation will be $\frac{y_{n+1}-y_n}{\Delta t}$. As for the second-order derivative $\frac{d^2y}{dt^2}$, we can use forward difference method for two times and approximate this derivative as $\frac{y_{n+1}-2y_n+y_{n-1}}{(\Delta t)^2}$

If we then apply this method to pendulum models, we can get the numerical solutions for pendulum without air resistance as

$$\theta_{n+1} = \left(2 - \frac{gh^2}{L}\right)\theta_n - \theta_{n-1} \tag{7}$$

And the numerical solution for pendulum with air resistance as

$$\omega_{n+1} = \omega_n + \Delta t\left(-\frac{g}{L}\sin\theta_n - \frac{\beta}{mL}\omega_n\right). \tag{8}$$

$$\theta_{\mathrm{n+1}} = \theta_{\mathrm{n}} + \Delta\mathrm{t} \cdot \omega_{\mathrm{n}} \tag{9}$$

4.2. RK4 method

RK4 method is a comparatively more accurate method. It requires 4 the slope of four points and uses the weighted average to replace the derivative. For example, in order to discrete

$$\frac{dy}{dt}$$

= f(t,y), the RK4 method will calculate

$$\begin{cases} k_1 = f(t_n,\ y_n) \\ k_2 = f\left(t_n + \frac{\Delta t}{2},\ y_n + \frac{\Delta t}{2}k_1\right) \\ k_3 = f\left(t_n + \frac{\Delta t}{2},\ y_n + \frac{\Delta t}{2}k_2\right) \\ k_4 = f(t_n + \Delta t,\ y_n + \Delta t k_3) \\ k_{n+1} = y_n + \frac{\Delta t}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) \end{cases} \tag{10}$$

Applying this to the spring pendulum model, we can get a system of equations and iterative motion of the spring pendulum.

5. Results and analysis

5.1. Pendulum without air resistance

In the case of pendulum without air resistance, we use Python; specifically, we use numpy package to calculate and use matplolib to plot, to draw the figure of the change of angle over time. We find that the motion of the pendulum without air resistance is periodic. The maximum value of the angel and the minimum value of the angel are opposite numbers. We also draw an animation for this case with matplolib.animation package. However, we find that the figure of the analytical solutions (Figure 1)and the numerical solutions (Figure 2) are so similar that as the time interval is small, we cannot visually see the difference between the two figures and animations. Thus, we further compare the numerical solution and the analytical solution later in order to see the difference.

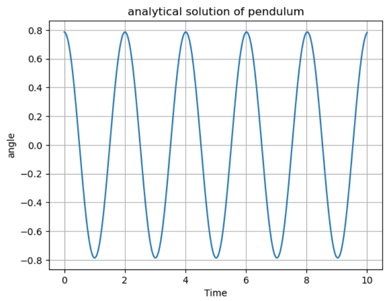

Figure 1. Analytical solutions for pendulum without air resistance

5.2. Pendulum with air resistance

As for the pendulum with air resistance, we also draw a figure (Figure 3) to show the change of the angle over time and the animation with the same tool that we used to visualize the pendulum without air resistance. We find that the amplitude of

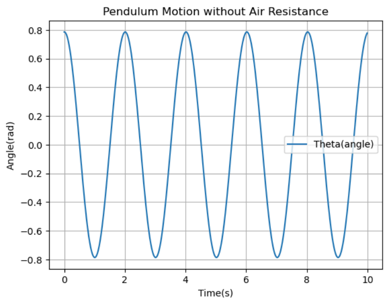

Figure 2. Numerical solutions for pendulum without air resistance

The function for the angel decreases over time and finally it converges to 0. This is reasonable since the air resistance will slow the pendulum down and finally let it stop. Moreover. The period of the change in the angle increases slightly over time because of the air resistance.

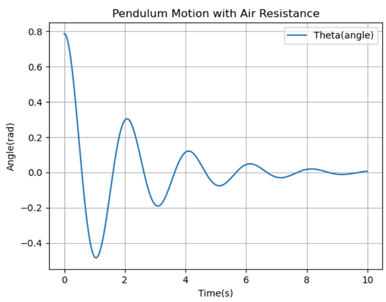

Figure 3. Numerical solutions for pendulum with air resistace

5.3. Spring pendulum

The motion of the spring pendulum is much more complex. The motion of it is not periodic. That's because there are two coupling equations in this system, which means that the motion of the spring

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and the motion of the pendulum as it swings will interact and cause the motion of the whole system to be in a mess. We also use Python to draw the figure of the change in angel over time (Figure 4) and the animation of the spring pendulum.

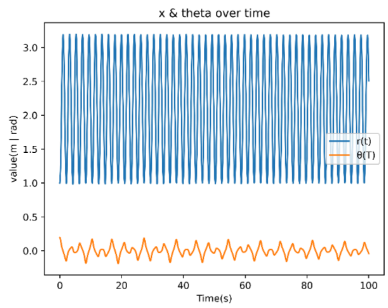

Figure 4. Numerical solutions for spring pendulum

6. Comparison

This paper makes two sets of comparisons: numerical solution of the pendulum without air resistance vs. the analytical solution of the pendulum without air resistance and numerical solutions of the pendulum without air resistance vs. the pendulum with air resistance. As for the first comparison, we expect that there will be an increasing difference between analytical solutions and numerical solutions since there is a truncation error between them and this error will add up over time and finally make a big difference. However, after plotting the figure of the difference between analytical solutions and numerical solutions (Figure 5), we find that the difference between them first becomes smaller, and then after it gets a certain value, the difference begins to grow. Initially, it confuses us. After our further consideration, we think that this situation is due to the fact that there is not only one error in our modeling. There are errors, including approximation error, truncation error, and even the accuracy of the computer in our model. These errors might counteract at first, which engenders the reduction of the difference at first. And then, the truncation error becomes dominant since it adds up, so the difference becomes bigger and bigger. The second comparison, which compares the difference between the pendulum with and without air resistance (Figure 6), fits our expectation. The difference between them will become smaller and smaller and finally, the function to show the difference will be periodic since the pendulum with air resistance will stop at a certain time step. After the pendulum with air resistance stops, the difference between the angel of the pendulum with air resistance and of the pendulum without air resistance can be seen as the difference between the angel of the pendulum without air resistance and 0. Thus, the function for showing the difference between this set of comparisons will finally be periodic.

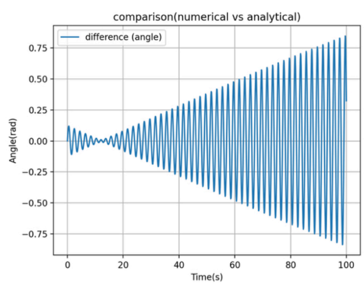

Figure 5. Comparison between analytical solutions and numerical solutions

DOI: 10.54254/2753-8818/2026.HZ31084

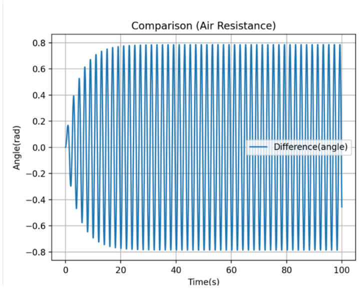

Figure 6. Comparison between pendulum with and without air resistance

7. Sensitivity test

7.1. Sensitivity test for pendulum without air resistance

In this research we did 3 set of sensitivity test for pendulum without air resistance, including calculating the difference when increasing and decreasing the length of the rope (Figure 7), the length of the gravitational parameter (Figure 8), and degree of initial angle (Figure 9), and plotted a graph with 3 lines representing increase, decrease and original for each set of sensitivity test. By analyzing the result of the calculation and the graph, we found that increasing the length will cause the period of the pendulum's angle increase. On the other hand, when increasing the length of the gravitational parameter, the period of the pendulum's angle will decrease. The increase in the degree of the initial angle will cause the amplitude of the pendulum's angle to increase. The result we found by the sensitivity test fits with real-life scenarios. For example, when a swing is released to a greater degree, it will swing higher (larger maximum angle). A pendulum clock with a longer length will move more slowly than a pendulum clock with short length, showing the increase in initial rod length will increase the period. Also, the difference speed of pendulum movement on different planet shows the effect of the gravitational parameter's effect on the pendulum's period.

7.2. Sensitivity test for spring pendulum

For spring pendulum, we did 3 set of sensitivity test, including calculating the difference when increasing and decreasing the spring constant (Figure 10), the degree of initial angle (Figure 11), and the rod's initial length (Figure 12), and plotted a graph with 2 lines, one representing the rod, and one representing the angle, for the control group, the spring pendulum with bigger K, the spring pendulum with smaller initial angle, and the spring pendulum with smaller initial length.

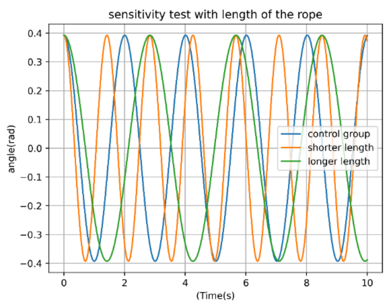

Figure 7. Sensitivity test for pendulum's rod length

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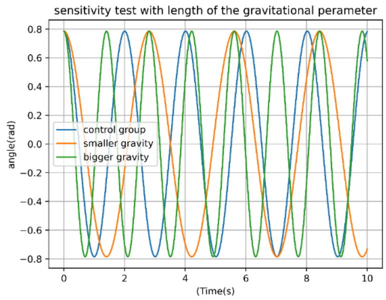

Figure 8. Sensitivity test for pendulum's gravitational parameter

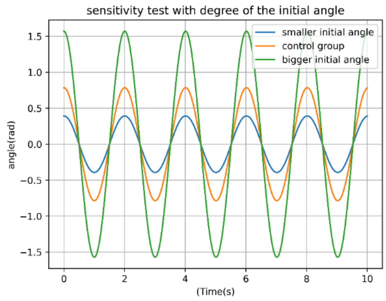

Figure 9. Sensitivity test for pendulum's initial angle

By comparing the 3 graphs with the graph of the control group, we found that when the spring constant increases, the frequency of the rod increases and the amplitude of the rod decreases; when the degree of initial angle increases, the amplitude of angle increases; when the rod's initial length increases, the amplitude of both the rod and angle increases. The result got from this sensitivity test can also fit in to real life scenarios. The bungee cord is a spring pendulum structure, and its movement can fit with all the results for the spring pendulum sensitivity test. A stiffer bungee cord (bigger K) can move up and down faster, but it will have a smaller maximum distance. A bungee cord that is released in a higher place (larger initial angle) will have a larger maximum swinging angle. A longer bungee cord (longer initial rod length) will swing higher and reach a lower position (larger angle and rod amplitude).

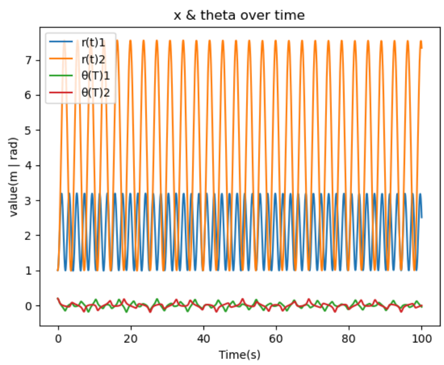

Figure 10. Sensitivity test for spring coefficient

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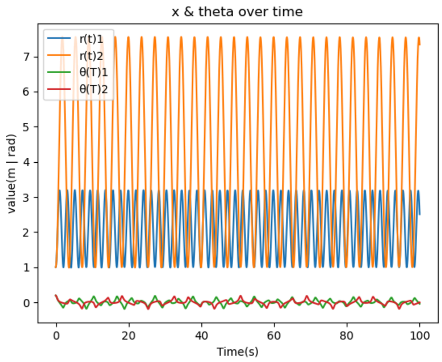

Figure 11. Sensitivity test for the spring pendulum's initial angle

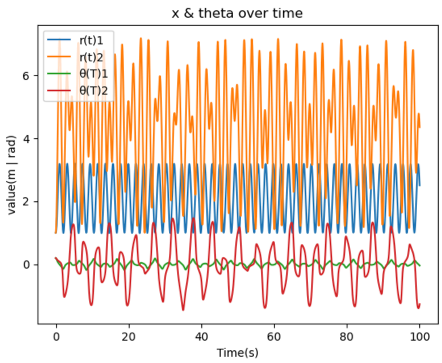

Figure 12. Sensitivity test for the pendulum's spring's length

8. Conclusion

By constructing several different mathematical models, this research had analyzed the motion of a pendulum with air resistance, a pendulum without air resistance, and a spring pendulum. By comparing the motion of different kinds of pendulums, and the change in factors that will affect the pendulum, we got our final results. The air resistance had a significant effect on the motion of a simple pendulum, which caused the pendulum's swing angle to decrease over time. Also, changes in the gravitational parameter, initial angle, and initial length can affect the pendulum angle's amplitude and period. For the spring pendulum, which is found to have a more complex and non-periodic motion, the changes in spring constant, initial angle, and initial rod length can change its angle, period and amplitude.

In the future, the results and model got from this research can also be applied in other fields and in real life. For example, the motion of a spring pendulum can be applied in analyzing the motion of human limbs, since muscles have a similar function as spring, and the movement of limbs is similar to that of a pendulum. The application of the result of this research in to real life can have a significant effect, since it can help to solve many kinds of real-life problems, and related real-life systems can be easily constructed.

At last, the research can still be improved, since it only analyzed some basic factors on pendulum motion. For the spring pendulum, for example, since its motion are too complex, the research didn't have a deep analyze on its motion.

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Author contribution

Jinchao Ma, Haotian Pan, and Jun Cao contributed equally to this work and should be considered co-first authors.

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