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A Concise Examination of the Riemann Zeta Function Drawn from An Invitation to Modern Number Theory

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Abstract. The Riemann zeta function $\zeta(s)$ occupies a fundamental role in analytic number theory. This review covers its primary attributes, initiating with the analytic extension past the convergence area $\Re(s) > 1$. The functional equation receives special focus, illustrating the function's symmetry and its evaluations at distinct locations, particularly negative integers. The study delves into the association of $\zeta(s)$ with the theta function through the Mellin transform, revealing connections to modular forms. Representation theory from the Heisenberg group further elucidates the zeta function's spectral and harmonic elements in automorphic forms. The examination concludes with uses in prime number theory, demonstrating $\zeta(s)$'s contribution to detailed prime distribution models. These aspects together highlight the zeta function's extensive relevance in number theory and associated areas.

Keywords: Riemann zeta function; analytic continuation; modular forms; prime distribution

1. Introduction

The Riemann zeta function arises naturally from attempts to encode arithmetic information into analytic objects. It is initially defined through a Dirichlet series that converges in a suitable region of the $\mathbb{C}$-plane, and its extension beyond this region reveals a remarkably rich structure [1]. Riemann's 1859 memoir (1) marked a turning point by demonstrating that the properties are intimately related to the density of primes.

A primary theme in the theory is the zeros. While trivial zeros are well understood, the nontrivial zeros lying within the critical strip exert a profound influence on arithmetic phenomena [2,3]. The Riemann hypothesis asserts a symmetry in their location, proposing that all such zeros lie on a single vertical line. Although unproven, this conjecture has shaped much of modern research in analytic number theory [4].

From a broader perspective, the zeta function extends its influence beyond number theory. It appears in diverse mathematical and physical frameworks, including probabilistic models, spectral theory, and quantum systems. Analytically, its special values encode classical invariants such as Bernoulli numbers, while its meromorphic continuation exhibits both a simple pole and a highly nontrivial zero structure.

The relation between the zeta function and primes becomes particularly transparent via Euler's product expansion (9), which expresses the function in terms of contributions from individual primes

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[5]. This viewpoint forms the analytic foundation of the prime theory, which is proved independently by Hadamard and de la Vallée Poussin (2), where the absence of zeros on a certain vertical line plays a decisive role. More refined explicit formulas further show that deviations in the density of primes are governed by the nontrivial zeros [6], illustrating the deep interaction between complex analysis and arithmetic.

2. Preliminaries

Following the outline of the Riemann zeta function's role in arithmetic and prime distribution, the next step outlines the supporting framework. Here, essential concepts are revisited, including the Euler product expansion [7], Mellin transform methods [8], and modular group properties [9]. These serve as the structural basis for the analytic and algebraic approaches in later sections.

Definition 1 (Euler product (9)). *The ζ admits both a Dirichlet series representation and the product expansion. The latter expresses zeta as an infinite product indexed by prime numbers, revealing the fundamental connection between its analytic properties and the distribution of primes.*

Definition 2 (Fourier transform). *The transform associated to a new function in the frequency domain. This transform plays a central role in harmonic analysis by encoding global information about a function in terms of its oscillatory components.*

Theorem 1 (Fourier series). *Any function that is periodic with a fixed period can be expanded into a series of complex exponentials. This representation decomposes the function into discrete frequency modes, with coefficients determined by averaging over a single period.*

Definition 3 (Modular group (6)). *The modular group is obtained from the group of integer matrices with determinant one by identifying matrices that differ by a sign. This group acts naturally on the imaginary-positive complex plane and serves as a fundamental object in the modular forms and geometry.*

Definition 4 (Mellin transform).

$$M[f](s) = \int_0^\infty f(x)t^{s-1}\,dx. \tag{1}$$

Definition 5 (Regularization). *To handle divergent integrals of the form $\int_0^\infty f(t)\,dt$, exponential regularization is applied by adding a decay factor $e^{-\lambda t}$ with $\lambda > 0$, obtaining*

$$I(\lambda) := \int_0^\infty f(t)e^{-\lambda t}\,dt. \tag{2}$$

The regularized integral is interpreted via the limit Reg$(I) := \lim_{\lambda\to 0^+} I(\lambda)$, if it exists or can be analytically continued.

3. Theta Function and Special Values of ζ at Negative Integers

With the foundational elements in place, the theta function is examined next, a crucial tool in analytic number theory [10]. Due to its analytic structure and transformation under modular substitutions, it acts as a potent means for extending ζ beyond its original domain. This approach provides a precise description of $\zeta(s)$ at negative integers, showing their arithmetic significance and connection to Bernoulli numbers [11]. For completeness, we collect here several classical analytic identities that are used throughout the paper.

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3.1. Modular Behavior of the Theta Function

Building on the lemmas, the modular properties of the theta function are investigated. Poisson summation and Gaussian integrals derive its transformation identity, revealing core symmetry. The theta function is defined as

$$\theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}, \tag{3}$$

satisfying $\theta(x) = \theta(x+2)$ and $\theta(1/x) = \sqrt{x}\cdot\theta(x)$.

3.2. Transform of the Theta Function

The modular change of the theta inspires the integral transforms as a powerful analytical tool [12]. In particular, applying the Mellin transform provides an explicit link between the theta and the zeta function, expressed through a central identity. Let$\phi(t) = \theta(t) - 1 = 2\sum_{n=1}^{\infty} e^{-\pi n^2 t}$.Applying the Mellin transform (Equation (4)), we compute

Proof. The Mellin transform of $\phi(t)$ is

$$M(s) = \int_0^\infty \phi(t)t^{s-1}\,dt = 2\sum_{n=1}^{\infty}\int_0^\infty e^{-\pi n^2 t}t^{s-1}\,dt. \tag{4}$$

To evaluate the inner integral, substitute $u = \pi n^2 t$, so $t = u/(\pi n^2)$ and $dt = du/(\pi n^2)$. The integral becomes

$$\int_0^\infty e^{-u}\left(\frac{u}{\pi n^2}\right)^{s-1}\frac{du}{\pi n^2} = \frac{1}{\pi^s n^{2s}}\int_0^\infty e^{-u}u^{s-1}\,du = \frac{\Gamma(s)}{\pi^s n^{2s}}, \tag{5}$$

since $\int_0^\infty e^{-u}u^{s-1}\,du = \Gamma(s)$ for $\Re(s) > 0$. Summing over n,

$$M(s) = 2\sum_{n=1}^{\infty}\frac{\Gamma(s)}{\pi^s n^{2s}} = 2\pi^{-s}\Gamma(s)\sum_{n=1}^{\infty}n^{-2s} = 2\pi^{-s}\Gamma(s)\zeta(2s). \tag{6}$$

Thus, $M(s) = 2\pi^{-s}\Gamma(s)\zeta(2s)$.

In other words,

$$\int_0^\infty (\theta(t)-1)t^{s-1}\,dt = 2\pi^{-s}\Gamma(s)\zeta(2s). \tag{7}$$

3.3. Symmetry of the Mellin Transform

The modular transformations of the theta function, analyzed through the Mellin transform, reveal a profound internal symmetry [13]. This approach shows how diverse analytic and algebraic methods integrate into a cohesive framework, linking exponential sums with the Riemann zeta function [14]. Let $\theta(t) = \phi(t) + 1$, so

$$\phi(t) = \frac{1}{\sqrt{t}}\phi\left(\frac{1}{t}\right) + \frac{1}{\sqrt{t}} - 1. \tag{8}$$

The Mellin transform becomes

$$M(s) = \int_0^\infty \left(\frac{1}{\sqrt{t}}\phi\left(\frac{1}{t}\right) + \frac{1}{\sqrt{t}} - 1\right)t^{s-1},dt \quad = \int_0^\infty t^{s-3/2}\phi\left(\frac{1}{t}\right),dt + \int_0^\infty t^{s-3/2},dt - \int_0^\infty t^{s-1},dt.$$

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Substitute $t = 1/u$, $dt = -du/u^2$, yielding

$$\int_0^\infty t^{s-3/2}\phi\left(\frac{1}{t}\right)\,dt = \int_0^\infty u^{1/2-s}\phi(u)\,du = M\left(\frac{1}{2}-s\right). \tag{9}$$

Thus, the symmetry relation is

$$M(s) = M\left(\frac{1}{2}-s\right) + I(s), \tag{10}$$

where

$$I(\lambda) = \int_0^\infty t^{s-3/2}e^{-\lambda t}\,dt - \int_0^\infty t^{s-1}e^{-\lambda t}\,dt = \lambda^{-(s-1/2)}\Gamma\left(s-\frac{1}{2}\right) - \lambda^{-s}\Gamma(s). \tag{11}$$

For $s < 0$, $\lim_{\lambda\to 0^+} I(\lambda) = 0$, so

$$M(s) = M\left(\frac{1}{2}-s\right). \tag{12}$$

This implies the functional equation for the zeta function:

$$\pi^{-x}\Gamma(x)\zeta(2x) = \pi^{-(1/2-x)}\Gamma\left(\frac{1}{2}-x\right)\zeta(1-2x). \tag{13}$$

This equation captures the theta function's symmetry and its link to the zeta function.

3.4. Analytic Continuation

Building upon the relations established in Section 3.3, we now extend the Riemann zeta function to negative integers through its analytic continuation [15]. This approach naturally reveals the well-known connection between $\zeta(s)$ and the Bernoulli numbers. The result can be expressed compactly as

$$(n+1)\zeta(-n) = -B_{n+1}, \qquad n \in \mathbb{Z}^+, \tag{14}$$

where B_n is the nth Bernoulli number. This identity encapsulates the deep interplay between the analytic behavior of $\zeta(s)$ and the arithmetic properties encoded in the Bernoulli sequence.

We begin by recalling three fundamental functional identities involving the Gamma and zeta functions:

$$\begin{cases} \Gamma(x)\Gamma(1-x) = \dfrac{\pi}{\sin(\pi x)}, & \text{(Reflection formula)}, \\ \Gamma(x)\Gamma\left(x+\frac{1}{2}\right) = \sqrt{\pi}\,2^{1-2x}\Gamma(2x), & \text{(Duplication formula)}, \\ \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), & \text{(Functional equation)}. \end{cases} \tag{15}$$

Substituting $x = \frac{s}{2}$ into the first identity yields

$$\Gamma\left(\tfrac{s}{2}\right)\Gamma\left(1-\tfrac{s}{2}\right) = \frac{\pi}{\sin(\frac{\pi s}{2})}. \tag{16}$$

Similarly, setting $x = \frac{1-s}{2}$ in the duplication formula gives

$$\Gamma\left(\tfrac{1-s}{2}\right)\Gamma\left(1-\tfrac{s}{2}\right) = \sqrt{\pi}\,2^{s}\Gamma(1-s). \tag{17}$$

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Substituting these two relations into the functional equation of $\zeta(s)$, we obtain

$$\begin{aligned}\frac{\zeta(s)}{\zeta(1-s)} &= \pi^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \\ &= \pi^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)\sin\left(\frac{\pi s}{2}\right)}{\pi} \\ &= \pi^{s-\frac{1}{2}} \frac{\sqrt{\pi}\, 2^s \Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)}{\pi}\end{aligned}$$

After simplification, this leads to the classical symmetric form

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\tfrac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s). \tag{18}$$

To determine the values at negative integers, let $s = -n$ with $n \in \mathbb{Z}^+$. Then

$$\zeta(-n) = 2^{-n}\pi^{-n-1} \sin\left(-\tfrac{\pi n}{2}\right) \Gamma(1+n)\zeta(1+n). \tag{19}$$

When n is odd, the sine term introduces a sign, and using the explicit evaluation of $\zeta(2k)$ from equation(30),we find

$$\zeta(-n) = (-1)^{\frac{n+1}{2}}(-1)^{\frac{n+1}{2}+1}\pi^{-n-1}2^{-n}n!\frac{(2\pi)^{n+1}B_{n+1}}{2(n+1)!} = -\frac{B_{n+1}}{n+1}. \tag{20}$$

When n is even, the sine term vanishes, implying that $\zeta(-n) = 0$. Hence, for all positive integers n,

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}. \tag{21}$$

This completes the analytic continuation of the Riemann zeta function to negative integers and establishes its deep relationship with the Bernoulli numbers.

4. Heisenberg Group and Zeta Function

The Heisenberg group is central in quantum mechanics and representation theory, connecting to zeta through analytic structures [16]. In the Schrödinger model, it acts on Gaussian spaces, linking to theta functions via Poisson summation. These underpin modular forms and zeta's functional equation [17]. Representation, harmonic analysis, and modular forms bridge the Heisenberg group to zeta [18].

4.1. Heisenberg Group

The Heisenberg group H_n is $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with operation

$$(a,b,c)\cdot(a',b',c') = \left(a+a', b+b', c+c'+\frac{1}{2}(a\cdot b' - b\cdot a')\right). \tag{22}$$

4.2. Schrödinger Representation

The representation on $L^2(\mathbb{R}^n)$ is

$$\rho(x,y,z)\psi(t) = e^{2\pi i(z+y\cdot t)}\psi(t+x). \tag{23}$$

For Gaussian $\phi(t) = e^{-\pi t^2}$, the theta function arises from discrete subgroup actions.

Compare with standard theta $\theta(\tau,t) = \sum_{n\in\mathbb{Z}} e^{2\pi i n t + \pi i n^2 \tau}$.

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5. The Riemann Zeta Function and the Prime Number Theorem

The interplay between analysis and number theory is shown through zeta, impacting prime distribution.

5.1. The Asymptotic Density of Square-Free Integers

The density of square-free integers can be derived using the Möbius function and the Euler product representation of the Riemann zeta function [19-21]. Since the Möbius function $\mu(n)$ vanishes precisely on integers divided by p^2, the quantity serves as an indicator of square-free integers. By summing this indicator and applying Möbius inversion, the counting problem is reduced to a main term governed by a convergent Euler product together with a lower-order error term [22,23]. Consequently, the proportion of square-free integers among all positive integers converges to $1/\zeta(2)$, which equals $6/\pi^2$.

5.2. Prime Number Theorem

To study the density of prime numbers, we introduce the $\pi(x)$, which is the number of primes not larger than x. For analytic purposes, it is convenient to consider

$$\psi(x) = \sum_{n \leq x} \Lambda(n), \tag{24}$$

where $\Lambda(n)$ is the von Mangoldt function, assigning $\log p$ to powers of primes and 0 otherwise.

The logarithmic derivative of the Riemann zeta function $\zeta(s)$ encodes prime powers in its Dirichlet series [24]. Using techniques from complex analysis, shifting the integration contour and applying the residue theorem, the main contribution comes from the simple pole $s = 1$, leading to the asymptotic estimate

$$\psi(x) \sim x \quad \text{as } x \to \infty.$$

From this, the classical prime theory follows:

$$\pi(x) \sim \frac{x}{\log x}.$$

5.3. Dirichlet L-Functions

To advance the analysis of primes in arithmetic progressions, we introduce L-functions [25]. Let χ be a Dirichlet character modulo q.

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}. \tag{25}$$

This series can also be expressed as a product over primes, which highlights how primes in certain residue classes contribute to the function. Analytic continuation and functional equations of $L(s, \chi)$ allow one to extend results similar to those for $\zeta(s)$ [26].

5.4. Primes in Arithmetic Progressions

Let p be a residue class modulo q. We define the weighted prime-counting function for this progression by

$$\psi(x; q, p) := \sum_{\substack{n \leq x \\ n \equiv p \pmod q}} \Lambda(n), \tag{26}$$

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where $\Lambda(n)$ denotes the von Mangoldt function.

Using an indicator function constructed from Dirichlet characters, one can decompose $\psi(x; q, p)$ into contributions from the principal character χ_0 and the non-principal characters. Analytic properties imply that the principal character dominates, giving

$$\psi(x; q, p) \sim \frac{x}{\varphi(q)} \quad \text{as } x \to \infty,$$

where $\varphi(q)$ is Euler's totient function.

From this, one can deduce the asymptotic prime distribution:

$$\pi(x; q, p) \sim \frac{1}{\varphi(q)} \int_2^x \frac{dt}{\log t} \sim \frac{x}{\varphi(q) \log x}.$$

6. Conclusion

This review has clarified the Riemann zeta function's role as a link in analytic number theory. Beginning with series and Euler product (1; 9), analytic continuation via Mellin and theta extends it meromorphically. The functional equation, from theta symmetry, derives negative integer values tied to Bernoulli numbers.

The Heisenberg group and Schrödinger representation frame theta modularity in harmonic context (11). This leads to arithmetic outcomes, like the Prime Number Theorem and primes in progressions via Dirichlet L-functions. Zeta's non-vanishing on $\Re(s) = 1$ bridges analytics to prime asymptotics.

In essence, the zeta function channels spectral, modular, and complex analysis insights into arithmetic truths (12; 21).

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Appendix

Lemma 1 (Poisson sum(3)). *Let f be a sufficiently smooth function with rapid decay at infinity.*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \hat{f}(m), \tag{27}$$

where $\hat{f}$ denotes the Fourier transform.

Lemma 2 (Gaussian integral). *For any $a > 0$, the Gaussian integral over the real line is given by*

$$\int_{-\infty}^{\infty} e^{-au^2}\, du = \sqrt{\frac{\pi}{a}}. \tag{28}$$

Lemma 3 (Duplication formula). *The Gamma function satisfies:*

$$\Gamma(x)\Gamma\left(x+\tfrac{1}{2}\right) = 2^{1-2x}\sqrt{\pi}\,\Gamma(2x). \tag{29}$$

Lemma 4 (Even values of the Riemann zeta function (12)). *For every positive integer n, one has*

$$\zeta(2n) = \frac{(-1)^{n+1}(2\pi)^{2n}B_{2n}}{2(2n)!}, \tag{30}$$

where B_{2n} denotes the Bernoulli numbers.

Lemma 5 (Weierstrass product for the Gamma function). *$x \in \mathbb{R}^+$, the Γ admits the product representation*

$$\Gamma(x) = e^{-\gamma x}\frac{1}{x}\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)^{-1} e^{x/n}, \tag{31}$$

where γ is Euler's constant.

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Lemma 6 (Reflection formula). *The Γ holds for the functional identity*

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}. \tag{32}$$

Lemma 7 (Stone–von Neumann theorem (4)). *Unitary irreducible representations of Heisenberg with specific center action are isomorphic to the Schrödinger representation.*

Lemma 8 (Perron-Type Integral Representation (9)). *For arithmetic $a(n)$ with Dirichlet series $F(s)$, the sum is recovered via contour integral.*

Lemma 9 (Residue Theorem in Complex Analysis (20)). *Contour integrals equal sum of residues.*