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Analysis of Dark Matter Decoupling Mechanism:Theoretical Derivation and Numerical Calculation from Thermal Equilibrium to Relic Abundance

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Abstract. Decoupling of dark matter can be understood through the Boltzmann equation and calculate how much dark matter will eventually remain in the universe. In the course of our study, the complete Boltzmann equation can be used, but given the more uniform distribution of matter in the early universe, some less important factors are ignored, such as the movement of particles in space and weak external forces. By changing the variables, Boltzmann equation can be turned into the Likati equation, which describes the strength and equilibrium properties of dark matter interactions. Through numerical calculations and approximation analysis, dark matter still maintains a thermal equilibrium when the temperature of the universe is high. But as the universe slowly cools, the rate of dark matter interaction does not keep up with the rate of the expansion of the universe, so that dark matter is separated from the thermal equilibrium, and eventually forms a stable residual amount of dark matter. The research linked the Boltzmann equation to the detachment process of dark matter, giving a reliable way to calculate the remaining amount of dark matter. However, there are some shortcomings in this study, such as assuming that the universe is completely homogeneous, considering only one way of interacting with dark matter. In later studies, we intend to add more real cosmic conditions and experimental data to improve the model, so that the nature of dark matter can be more accurately determined.

Keywords: Decoupling of dark matter, Boltzmann equation, Cosmology

1. Introduction

In the field of astrophysics, dark matter is a particularly mysterious substance, accounting for approximately 27% of the total mass and energy of the universe. This proportion is much larger than the 5% that ordinary matter makes up. Although dark matter does not participate in electromagnetic interactions and cannot be directly detected by traditional optical or radio methods, its gravitational effect plays a crucial role in the formation of large-scale cosmic structures, in the rotation curves of galaxies, and in the anisotropy of cosmic microwave background radiation [1].

The phenomenon of dark matter decoupling is a crucial aspect for understanding the development of the universe and the distribution of matter. It describes how the interaction between dark matter

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and other matter in the universe gradually weakens and how it ”separates” from thermal equilibrium [2].

Theoretically speaking, the Boltzmann equation is the fundamental equation in statistical physics used to describe the dynamic behavior of gases, which links the statistical behavior of microscopic particles with macroscopic transport phenomena. It has a significant influence in both theoretical physics and engineering applications. When conducting research on the decoupling of dark matter, the theoretical framework established based on the Boltzmann equation can explore from a microscopic perspective how the distribution function of dark matter particles changes over time and space during the evolution of the universe. This, in turn, can help us discover the physical mechanism and fundamental laws of dark matter decoupling.

By studying the decoupling of dark matter, we can calculate the residual abundance of dark matter more precisely, which provides the necessary parameters for cosmological models and enables us to have a deeper understanding of the early stages of cosmic evolution and matter distribution. This research can solve the problem of the formation of cosmic structures and explain how large-scale structures such as galaxies and galaxy clusters are generated and developed It also provides a new approach for the exploration of particle physics. It enables us to study the properties and interactions of dark matter particles and to seek new physics beyond the Standard Model [2].

It is highly innovative and necessary to conduct research on dark matter decoupling starting from the Boltzmann equation, as it holds promise to provide new ideas and methods for solving the mystery of dark matter.

2. Introduction of boltzmann equation

In the discipline of thermodynamics, the Boltzmann equations are particularly important, explaining how the phenomena of heat transfer and diffusion occur from a microscopic point of view. This equation tells us that at the molecular level, why heat travels from the high temperature to the low temperature, and how they diffuse when the pressure on both sides of the gas is different. In statistical physics, this equation is like a bridge, which links the motion of individual molecules to the thermodynamic properties of the whole system [3].

The equation can completely describe the statistical behavior of all particles in the non-equilibrium thermodynamic system. It takes into account the motion of particles, the influence of external forces on particles, and the collision between particles, and the factors that collide between particles, and it tells us exactly how the distribution function of the particles changes with time and position. From a microscopic point of view, this equation reflects the motion of each particle in the space of the phase and the interaction between them. For example, by solving this equation, we can figure out exactly what the gas’s thermal conductivity and diffusion coefficient are. [4] These calculations are particularly helpful for us to understand physical processes in the real world.

2.1. Recall the complete boltzmann equation

The complete Boltzmann equation is written as:

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{r}} f + \frac{\boldsymbol{F}}{m} \cdot \nabla_{\boldsymbol{p}} f = C(f, f)$$

$\frac{\partial f}{\partial t}$: How the particle distribution function $f(\boldsymbol{r}, \boldsymbol{p}, t)$ changes with time at a fixed position and momentum.

$\boldsymbol{v} \cdot \nabla_{\boldsymbol{r}} f$: How the distribution changes because particles move through space . Here, $\boldsymbol{v}$ is the particle velocity , and $\nabla_{\boldsymbol{r}}$ is the spatial gradient operator.

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$\frac{\boldsymbol{F}}{m} \cdot \nabla_{\boldsymbol{p}} f$: How the distribution changes because particles' momentum changes due to external forces $\boldsymbol{F}$. $\nabla_{\boldsymbol{p}}$ is the momentum gradient operator.

$C(f, f)$: The collision term, describing how particle collisions affect the distribution. It's the key part for how a system goes from non-equilibrium to equilibrium [1].

We can simplify the equation when collisions are the main factor changing the particle distribution. This happens under these conditions: First: frequent collisions or low particle density: Collisions happen so often that their effect on the distribution is much stronger than the effects of particles moving in space or changing momentum. Second: Weak or no external force: If the external force $\boldsymbol{F}$ is very weak or zero, the term $\frac{\boldsymbol{F}}{m} \cdot \nabla_{\boldsymbol{p}} f$ is nearly zero because momentum drift is negligible. Third: Spatial uniformity: If the system is very uniform in space , then $\nabla_{\boldsymbol{r}} f$ is nearly zero, so $\boldsymbol{v} \cdot \nabla_{\boldsymbol{r}} f$ is also nearly zero .

In situations where collisions dominate, the effects of spatial drift and momentum drift are much smaller than the collision effect. So we can approximate:

$$\boldsymbol{v} \cdot \nabla_{\boldsymbol{r}} f \approx 0, \quad \frac{\boldsymbol{F}}{m} \cdot \nabla_{\boldsymbol{p}} f \approx 0$$

Substituting these approximations into the complete Boltzmann equation, we get:

$$\frac{\partial f}{\partial t} \approx C(f, f)$$

If we consider the total rate of change of the distribution function with time as $\frac{df}{dt}$, the simplified equation becomes:

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial t} \right)_{\text{coll}}$$

The simplified form of the Boltzmann equation focuses on the dominant role of collisions in changing the particle distribution. We ignore the less important effects of particles drifting in space or changing momentum. This simplification works well when collisions are very frequent and other effects are negligible, like in some cases of rarefied gases or systems with strong collisions.

2.2. Simplifying the equation

When studying dark matter in the early universe, we can make some simplifications based on the universe's properties:

First: Uniformity and isotropy. The early universe is roughly the same in all places and directions. This means the distribution function f only depends on momentum size p and time t , so $f = f(p, t)$.

Second: Expansion effect. The universe is expanding, which makes particle momentum decrease over time. This change follows $\frac{dp}{dt} = -pH$, where H is the Hubble parameter .

With these, the total change of the distribution function simplifies to:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} - pH \frac{\partial f}{\partial p}$$

For dark matter particles X and their antiparticles $\overline{X}$, the main interaction is annihilation and the reverse process (light particles combining to form X and $\overline{X}$).

The collision term for X particles accounts for both processes: First: When X and $\overline{X}$ annihilate, the number of X particles decreases. Second: When light particles form X and $\overline{X}$, the number of X particles increases.

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Mathematically, this is written as an integral involving the distribution functions of all particles involved, ensuring energy and momentum are conserved during interactions.

The ”number density” n_X is found by integrating the distribution function over all possible momenta:

$$n_X(t) = \int \frac{d^3p}{(2\pi)^3} f_X(p,t)$$

When the system is in ”thermal equilibrium” (interactions happen fast enough to keep particle numbers stable), the distribution function follows a standard form (Boltzmann distribution). At equilibrium, the number of X particles created equals the number annihilated, so $n_X n_{\overline{X}} = (n_X^{\text{eq}})^2$.

By integrating the full Boltzmann equation over all momenta and using the equilibrium condition, we get an equation for how n_X changes with time:

$$\frac{dn_X}{dt} + 3Hn_X = -\langle \sigma v \rangle \left(n_X^2 - (n_X^{\text{eq}})^2 \right)$$

The left side includes the direct change of n_X and a term $3Hn_X$ to account for the universe expanding. The right side describes interactions: $\langle \sigma v \rangle$ is a measure of interaction strength , and the term $(n_X^2 - (n_X^{\text{eq}})^2)$ tells us if particles are more (or less than equilibrium—driving annihilation or creation.

To make the equation easier to work with, we use a dimensionless number density $N_X = \frac{n_X}{s}$, where s is the ”entropy density” . Substituting this in, we get the final form used for dark matter freeze-out:

$$\frac{dN_X}{dt} = -s\langle \sigma v \rangle \left(N_X^2 - (N_X^{\text{eq}})^2 \right)$$

This equation tracks how the relative number of dark matter particles changes over time, capturing the key transition from equilibrium to freeze-out.

First, we have a dimensionless number density N_X, which is defined as the physical number density n_X divided by the entropy density s. So:

$$N_X = \frac{n_X}{s}$$

This means the physical number density n_X is $n_X = sN_X$, and the equilibrium physical number density n_X^{eq} is $n_X^{\text{eq}} = sN_X^{\text{eq}}$.

2.3. Account for universe expansion

In an expanding universe, the physical number density n_X changes not just because of particle interactions but also because the universe is getting bigger, which dilutes the particles. The basic equation for how n_X changes with time is:

$$\frac{dn_X}{dt} + 3Hn_X = C(n_X)$$

Here, $\frac{dn_X}{dt}$ is the rate of change of n_X by itself, $3Hn_X$ accounts for the dilution due to expansion , and $C(n_X)$ is the term for particle interactions (like annihilation or creation).[1]

The formula bellow shows how N_X changes with time from the dark matter evolution equation:

$$\frac{dN_X}{dt} = -s\langle \sigma v \rangle \left(N_X^2 - (N_X^{\text{eq}})^2 \right)$$

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Now, take the derivative of $n_X = sN_X$ with respect to time:

$$\frac{dn_X}{dt} = s\frac{dN_X}{dt} + N_X\frac{ds}{dt}$$

In cosmology, the entropy density s is almost conserved, meaning $\frac{ds}{dt} + 3Hs = 0$, so $\frac{ds}{dt} = -3Hs$. Substitute $\frac{dN_X}{dt}$ and $\frac{ds}{dt}$ into the above equation:

$$\begin{aligned}\frac{dn_X}{dt} &= s\left[-s\langle\sigma v\rangle\left(N_X^2 - (N_X^{\text{eq}})^2\right)\right] + N_X(-3Hs)\\ &= -s^2\langle\sigma v\rangle\left(N_X^2 - (N_X^{\text{eq}})^2\right) - 3HsN_X\end{aligned}$$

Then substitute $N_X = \frac{n_X}{s}$ and $N_X^{\text{eq}} = \frac{n_X^{\text{eq}}}{s}$:

$$\begin{aligned}\frac{dn_X}{dt} &= -s^2\langle\sigma v\rangle\left(\left(\frac{n_X}{s}\right)^2 - \left(\frac{n_X^{\text{eq}}}{s}\right)^2\right) - 3Hn_X\\ &= -\langle\sigma v\rangle\left(n_X^2 - (n_X^{\text{eq}})^2\right) - 3Hn_X\end{aligned}$$

Comparing with the basic equation $\frac{dn_X}{dt} + 3Hn_X = C(n_X)$, we find the interaction term $C(n_X) = -\langle\sigma v\rangle\left(n_X^2 - (n_X^{\text{eq}})^2\right)$.

For a general particle reaction like $1 + 2 \leftrightarrow 3 + 4$ (particle 1 interacts with particle 2 to make particles 3 and 4, or vice versa), let $n_1 = n_X$, n_2 be the number density of another particle , and n_3, n_4 be the products. Let $\alpha = \langle\sigma v\rangle$ and β . Finally, we can write:

$$\frac{1}{a^3}\frac{d(n_1 a^3)}{dt} = -\alpha n_1 n_2 + \beta n_3 n_4$$

3. Dark matter decoupling and the boltzmann equation

Dark matter is a mysterious substance that makes up a large part of the universe's mass but doesn't interact with light. One important idea about how dark matter formed in the early universe is called "freeze - out"[5].

3.1. Basic reaction and assumptions

Imagine a heavy dark matter particle X and its antiparticle $\overline{X}$. They can annihilate each other to produce two light particles l and $\bar{l}$, like this:

$$X + \overline{X} \leftrightarrow l + \bar{l}$$

some simple assumptions are made: The light particles l are tightly connected to the rest of the hot soup in the early universe, so they always stay in thermal equilibrium. There's no initial difference between the number of X and $\overline{X}$, so their number densities are equal ($n_X = n_{\overline{X}}$). - Only the reaction $X + \overline{X} \leftrightarrow l + \bar{l}$ affects the number of X particles during freeze - out, and the temperature T of the universe is related to the scale factor a by $T \propto \frac{1}{a}$.

To describe how the number of X particles changes, we use the Boltzmann equation. Since the universe is expanding, the number density of particles gets diluted. So we consider the number of particles in a "comoving volume" . Let $N_X = n_X a^3$ be the number of X particles in this comoving volume. The Boltzmann equation for N_X is:

$$\frac{dN_X}{dt} = -\langle\sigma v\rangle\left(n_X^2 - (n_X^{\text{eq}})^2\right)a^3$$

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Here, $\langle\sigma v\rangle$ is a measure of how strong the interaction is (it's the average of the cross - section σ and velocity v of the particles). n_X^{eq} is the equilibrium number density of X particles, which depends on temperature and is given by a distribution like the Maxwell - Boltzmann distribution, $n_X^{\text{eq}} \propto T^3 e^{-M_X/T}$.

If we substitute $N_X = n_X a^3$ into the equation, we get:

$$\frac{1}{a^3}\frac{d(n_X a^3)}{dt} = -\langle\sigma v\rangle \left(n_X^2 - (n_X^{\text{eq}})^2\right)$$

To make the equation easier to handle, we introduce a dimensionless number density $Y_X \equiv \frac{n_X}{T^3}$. Since $T \propto \frac{1}{a}$, $T^3 \propto \frac{1}{a^3}$, so Y_X is related to the number of particles in the comoving volume.

A new variable $x \equiv \frac{M_X}{T}$ is defined . By changing variables from time t to x, and using the relationship between the Hubble parameter H and temperature, the Boltzmann equation can be simplified to a Riccati equation:

$$\frac{dY_X}{dx} = -\frac{\lambda}{x^2}\left[Y_X^2 - (Y_X^{\text{eq}})^2\right]$$

Here, λ is a dimensionless coupling parameter, $\lambda \equiv \frac{M_X^3\langle\sigma v\rangle}{H(M_X)}$, and $H(M_X)$ is the Hubble parameter corresponding to the mass M_X.

3.2. Freeze - Out process

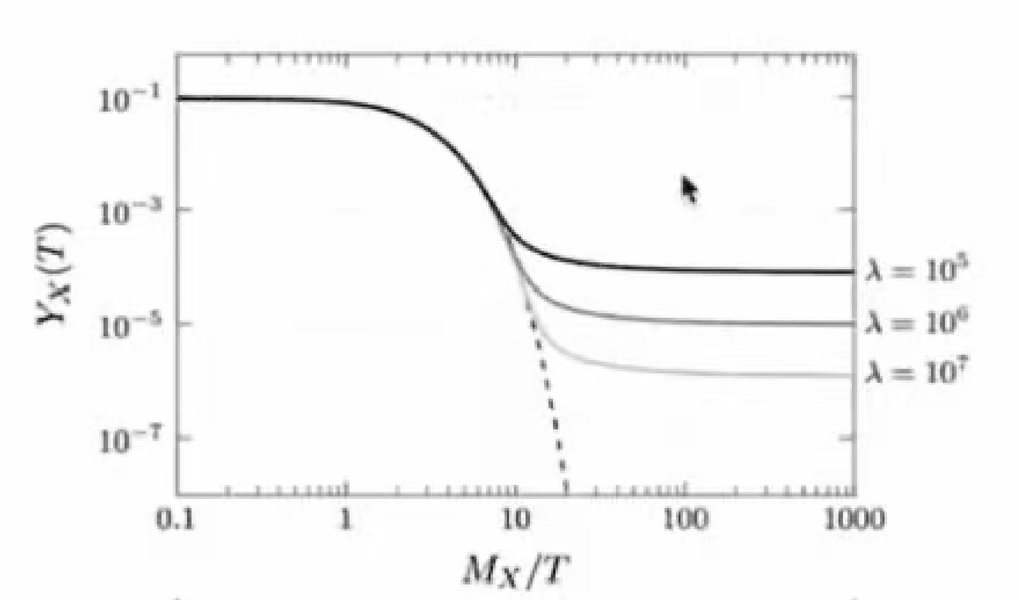

Figure 1: Plot of $Y_X(T)$ vs M_X/T for Different λ Values

Riccati equation numerically can be solved . The results show: At very high temperatures (small x, $T \gg M_X$), Y_X is almost equal to Y_X^{eq}, meaning X particles are in thermal equilibrium.[1] As the universe cools (x increases, T decreases), Y_X starts to deviate from Y_X^{eq} because the interaction rate becomes too slow to keep up with the expansion. When the universe is sufficiently cool (large x, $T \ll M_X$), the interactions are so rare that Y_X stops changing and becomes a constant Y_X^{∞}, which is called the ”relic abundance”.

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After freeze - out, the equilibrium term Y_X^{eq} is negligible (because $e^{-M_X/T}$ is very small). The Riccati equation simplifies to:

$$\frac{dY_X}{dx} \approx -\frac{\lambda}{x^2} Y_X^2$$

In Figure 1, Integrating this equation from the freeze - out time x_f (where $x_f \approx 10$ as an estimate) to $x \to \infty$, we get:

$$Y_X^\infty \approx \frac{x_f}{\lambda}$$

3.2.1. Number density today

To find the dark matter density today, we use the conservation of entropy. In the expanding universe, the entropy density s (related to T^3 and a factor g_* called effective degrees of freedom) is approximately conserved: $g_{*,s}(T)T^3a^3 =$ constant ($g_{*,s}$ is the effective degrees of freedom for entropy) [3].

The number density of dark matter today $n_{X,0}$ can be found using the relic abundance Y_X^∞, the today's temperature T_0, and the entropy conservation:

$$n_{X,0} = Y_X^\infty T_0^3 \frac{g_{*,s}(T_0)}{g_{*,s}(M_X)}$$

3.2.2. Dark matter density parameter

The energy density of dark matter today is $\rho_{X,0} = M_X n_{X,0}$. The critical energy density of the universe $\rho_{\text{crit},0}$ is related to the Hubble constant H_0. We define the dark matter density parameter Ω_X as the ratio of dark matter energy density to the critical energy density:

$$\Omega_X \equiv \frac{\rho_{X,0}}{\rho_{\text{crit},0}}$$

Substituting $n_{X,0}$ and using the approximation $Y_X^\infty \approx \frac{x_f}{\lambda}$, we can derive:

$$\Omega_X \sim 0.1 \frac{x_f}{\sqrt{g_*(M_X)}} \frac{10^{-8}\text{GeV}^{-2}}{\langle \sigma v \rangle}$$

If dark matter is a Weakly Interacting Massive Particle (WIMP), its $\langle \sigma v \rangle$ is of the order of the weak interaction cross - section. Plugging this into the formula for Ω_X, it matches the observed dark matter density very well. This coincidence is called the "WIMP miracle".

4. Conclusions and prospects

We used the Boltzmann equation to study this process and found some interesting results. First, it was started with this complete equation and did some simplified processing, such as ignoring the effects of particles in space and external weak forces.[1] Because the early universe was similar in all directions, an equation is got that describes how the number of dark matter particles changes over time. This equation explains how dark matter particles change from a thermobalance to a free state. The most amazing thing is that if dark matter is a particle like WIMP, the amount calculated is exactly the same as the density of dark matter, which is exactly the same as the observed density of dark matter, which is called the "WIMP miracle." [5] However, there are some shortcomings in this study. the early universe is completely uniform is assumed, and dark matter has only one way to interact, but

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it may actually be more complex. Other possibilities are not involved. [6] In future studies, models are improved and more real cosmic conditions are added that can more accurately predict when and ultimately the amount of dark matter is out of balance. [7] It can also be combined with experimental data from underground laboratories or particle colliders to directly measure the interaction intensity and mass of dark matter.

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