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# *Bijections Between Standard Young Tableaux and Classical Combinatorics Structures of Catalan Numbers*

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***Abstract.*** This paper surveys classic Catalan-counted structures and presents explicit bijections to standard young tableaux (SYT) of shape (n, n). We then construct concrete bijections linking SYT to Dyck paths, binary trees (and binary search trees), non-crossing partitions, stack-sortable permutations (via 231-avoidance), and covering antichains. Each construction highlights the same Catalan recursion from a different perspective and shows how combinatorial decomposition mirrors tableau constraints (row/column strictness). The goal is expository: to unify well-known Catalan objects through clean, reversible mappings that clarify why these families are equinumerous and how local rules (prefix conditions, subtree splits, bracket matching, etc.) correspond across domains. We conclude by summarizing the common schema behind these bijections and by indicating how such correspondences illuminate broader connections between enumerative combinatorics and the representation theory of the symmetric group.

***Keywords:*** Catalan numbers, standard Young tableaux, Dyck paths, binary trees, non-crossing partitions, stack-sortable permutations, 231-avoidance, bijection

## 1. Introduction

Catalan numbers are a famous sequence in mathematics that appears in many counting questions, from combinatorics to algebra, geometry, and even computer science. With applications to binary trees, polygon triangulations, Dyck paths, standard Young tableaux, and pattern-avoiding permutations, the Catalan sequence is often described as one of the most ubiquitous sequences in mathematics [1,2].

There is already over 200 years history of Catalan numbers. The person who first found it was the Chinese mathematician and scientist Ming Antu (1692—1763), who was from inner Mongolia. He wrote a book called Quick Method for Accurate Values of Circle Segments, which involves many trigonometric identities and power series, and some of them includes Catalan Number:

$$\sin(2\propto) = 2sin\propto - \sum_{n=1}^{\infty} \frac{C_{n-1}}{4^{n-1}} sin^{2n+1}\propto = 2sin\propto - sin^3\propto - \frac{1}{4} sin^5\propto - \frac{1}{8} sin^7\propto - \ldots.$$

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However, this series of number didn't play any role in the whole work. This book published in 1839 and after 149 years, the number sequence was finally found connecting with Catalan numbers by Luo Jianjin [2,3].

Within approximately the same number of decades, the Swiss polymath Leonard Euler introduced and found a closed formula which we call Catalan numbers now. With his teachers' and friends' help, he observed a pattern and guessed the following formula for Catalan numbers:

$$C_{n-2} = \frac{2 \bullet 6 \bullet 10 \cdots (4n-10)}{2 \bullet 3 \bullet 4 \cdots (\mathrm{n}-1)}$$

Euler's discovery was later formalized and generalized by Eugène Charles Catalan (1814–1894), who studied the same sequence in connection with parenthesized expressions and lattice paths. For his effort on studying the sequence, it was finally named by his name – Catalan numbers [4,5].

One important idea that connects all these different Catalan-number problems is the idea of a bijection. In combinatorics, a bijection is a one-to-one matching between two sets of objects. It shows that two different types of structures actually have the same number of elements for a deeper reason. For example, there are bijections between Dyck paths and certain standard Young tableaux, and also between non-crossing partitions and binary trees. These connections don't just prove the numbers are the same—they also help us understand how these objects share similar patterns or recursive rules.

This paper focuses on showing the beauty and usefulness of Catalan bijections. We are not trying to discover new formulas. Instead, we want to explain how different structures that are counted by Catalan numbers are related through clear and specific bijections. First, we will introduce some basic Catalan structures like Dyck paths and binary trees. Then, we will show how their bijections with Catalan numbers. Through these bijections, we hope to give a better understanding of why Catalan numbers appear so often and how they connect different areas of combinatorics.

## 2. Catalan numbers

Catalan numbers form an important sequence that appears in countless counting problems throughout combinatorics, geometry, computer science, and beyond. The sequence begins: $C_0 = 1,\ C_1 = 1,\ C_2 = 2,\ C_3 = 5,\ C_4 = 14,\ C_5 = 42,\ C_6 = 132,\ \ldots$

### 2.1. Closed-form formula

The n-th Catalan number can be defined using a closed-form formula:

$$C_n = \tfrac{1}{n+1}\binom{2n}{n} = \tfrac{(2n)!}{(n+1)!n!}$$

This formula expresses the number of ways to correctly match pairs of parentheses, structure binary trees, and construct Dyck paths, among many others. It grows rapidly with n, but much slower than the total number of permutations n!, which makes it especially interesting for counting restricted or structured objects.

### 2.2. Recursive relation

Catalan numbers also satisfy a beautiful recursive identity:

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$$C_0 = 1,\ C_n = \sum_{i=0}^{n-1} C_i \bullet C_{n-1-i}$$

This recurrence has a natural interpretation in several Catalan structures. For example, when counting binary trees, the sum represents all ways of choosing a left and right subtree of a root node such that the total number of nodes is n.

### 2.3. Combinatorial interpretations

Catalan numbers are famous for showing up in over 200 combinatorial structures. Some of the most well-known include:

• The number of valid ways to parenthesize an expression withn+1 factors
• The number of Dyck paths (lattice paths that never go below the x-axis)
• The number of binary trees with n internal nodes
• The number of non-crossing partitions of a set
• The number of stack-sortable permutations of length n
• The number of standard Young tableaux of shape (n, n)

Each of these interpretations leads naturally to a bijection, a one-to-one correspondence between structures, which is the central theme of this paper.

## 3. Group representation and Standard Young Tableaux

Group theory offers a framework for studying symmetry in algebraic and combinatorial contexts. The symmetric group $S_n$ , which encodes all permutations of n elements, is a particularly rich object whose structure has been extensively investigated [6]. To analyze such groups in a more concrete way, one method is representation theory, where abstract group elements are considered as linear transformations of vector spaces. This perspective not only provides powerful algebraic tools but also reveals deep connections with combinatorial structures [7].

An indispensable theme in this theory is the study of irreducible representations, which can be viewed as the indivisible building blocks of the representation category. For the symmetric groups, irreducible representations admit a strikingly combinatorial parametrization: they are indexed by integer partitions of n, or equivalently by Young diagrams [8].

Young diagrams, when filled with integers, producing Young Tableaux. These tableaux connect representation theory and combinatorics, encoding both algebraic and enumerative information. Among them, standard Young tableaux, tableaux with strictly increasing rows and columns, play a particularly distinguished role. They will appear naturally in the construction of irreducible representations, most notably in the Specht module approach [6,9].

For the object this paper researching about, the standard Young tableaux for the shape (n,n), which is a Young tableaux for size $2\times n$ . It also follows the rule that rows and columns are strictly increasing. For a given size of standard Young tableaux of shape (n,n), the number of ways to fill in the numbers is $C\_n$ , the $n^{th}$ Catalan number, [1].

One direct explanation about the number of ways for filling the standard Young tableaux being the $n^{th}$ Catalan number is constructing a bijection between correct parentheses sequences and standard Young tableaux:

Considering the properties of strictly increasing in rows and columns, this means the second row will never exceed the number of numbers in that of the first row. Assuming filling the Young diagram with numbers from 1 to 2n, and the diagram having numbers from 1 to 2k filled in the first

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(k,k) diagram, where 0<k<n. Then the next number to be filled is 2k+1, and it must be put in the first row, or otherwise no number could be satisfied to put in the first row smaller than 2k+1.

Then, this property can be naturally related to correct parentheses sequences, since both structures have a similar restriction: for correct parentheses sequence, at any point, the number of right parentheses can never exceed the number of left parentheses. This correspondence yields a direct bijection between standard Young tableaux of shape (n,n) and such sequences:

"(" $\leftrightarrow$Put the next number in the first row

")" $\leftrightarrow$Put the next number in the second row

To explain why the number of valid parenthesis sequences equals the n-th Catalan number, one may consider a recursive decomposition. Take a valid sequence of n pairs of parentheses and split it according to the position of the first closing parenthesis. This determines two independent subsequences, which we denote by A and B: A contains k pairs of parentheses for some $0 \leq k < n$; correspondingly, B contains the remaining $n-k-1$ pairs.

Since A and B can be chosen independently, the number of valid sequences satisfies the recurrence

$$C_n = \sum_{0}^{n-1} C_k C_{n-k-1}$$

is gained.

Therefore, the ways of standard young tableaux of shape (n,n) and correct parentheses sequence are being proved to be the $n^{th}$ Catalan number.

## 4. Bijection constructions

### 4.1. Bijection between Dyck path and Standard Young Tableaux

#### 4.1.1. Introduction to Dyck path

A Dyck path involves a staircase walk that start at the origin (0, 0), and stops at (2n, 0), including steps that grow (1, 1), called "U," and return (1, 1), called" D," with the condition that the path not also settles below the x-axis at any time.

A Dyck path of semilength n is a series of n up-steps and n down-steps so that any prefix corresponds to at least the sheer number of up-steps in the sequence.

For instance, UUDDUD, UUDUDUD, and UUDUDUD form valid Dyck paths of length 6 when n denotes 3. The Catalan number corresponds to the total number of Dyck paths with length 2n.

Dyck paths are fundamental to combinatorics since they have a recursive structure. Any Dyck path may be separated into an up-step, a down-step, and eventually a new Dyck path. Each peak, valley, and return to the x-axis shows a structural feature seen in other Catalan objects. In addition to mixed parentheses, binary tree shapes, and mountain ranges, Dyck paths contain a variety of combinatorial interpretations.

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### 4.1.2. Bijection to Standard Young Tableaux

This one-to-one correspondence established between the Dyck path and shape of (some integers, n) and the standard Young tableaux has become one of the most ingenious mappings in Catalan combinatorics. The construction of this mapping employs a clear algorithm. Label each step from 1 to 2n in order when given a Dyck path with a length of 2n. Point the up-step numbers from the first row of the tableau, and the down-step numbers in the second row, maintaining their positions in agreement with the path's order. The tableau meets the SYT rules: entries expand through columns and from side to side in rows because the path nowhere drops below the x-axis.

The Dyck path UUDUDD has steps 1 through 6, where the up-steps occur at positions 1, 2, 4, and 3, 5, and 6. This results in a tableau with first row: 1, 2, 4, and second row: 3, 5, 6, which happens to be a valid SYT of shape (3, 3).

This bijection is reversible. We use the numbers 1 through 2n employing a standard Young tableau of shape (n, n). We substitute a number in a first row with an up-step (U); in the second row, we use a down-step (D). Because the tableau guarantees correct orderings, this provides a Dyck path that sustains the prefix condition correctly.

## 4.2. Bijection between binary tree problems and Standard Young Tableaux

### 4.2.1. Introduction to binary tree

There are mainly two binary tree problems related to Catalan numbers: The number of binary tree structures with n nodes and the number of binary search trees with n given numbers.

A tree is a dynamic data structure consisting of nodes connected by edges, and A binary tree is a tree in which each node has at most two children (child means an immediate successor of a node). Moreover, a binary search tree is a binary tree satisfying that the left-hand child has a value less than its parent and right-hand child has a value greater than its parent. Examples of the three trees are shown in the Figure 1.

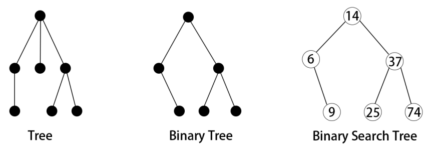


Figure 1. Examples of three kind of trees

For the first question: The number of binary tree structures with n nodes. Let $B_n$ denotes the number of binary tree structures with n nodes, then the binary tree could be decomposed into three parts, as shown in the Figure 2.

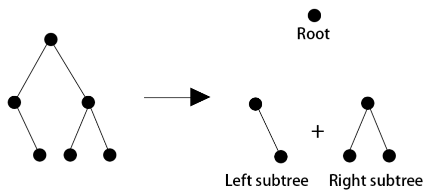


Figure 2. The process of decomposition from one binary tree to root, left subtree, and right subtree

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Let $B_{n,k}$ denotes the number of binary tree structures with n nodes that have a left subtree with k nodes, and this decomposition defines a relationship

$$B_{n,k}=B_k \times B_{n-k-1}$$

The size of k ranges from [0, n-1], and $k \in N_0$ ; thus

$$B_n=\sum_{k=0}^{n-1} B_{n,k}$$

Hence, the number $B_n$ of the binary tree with n nodes satisfies the recurrence equation

$$B_n=\sum_{k=0}^{n-1} B_k B_{n-k-1}$$

for n>1. Since $B_0=1$ , this is identical to the Catalan recurrence, and thus $B_n = C_n$ for all $n \geq 0$ .

For the second question, the number of binary search trees with n given numbers. If the root is the (k+1)-th large value, then there are k numbers in the left subtree and n-k-1 numbers in the right subtree. Through the same process, the number of binary search tree could be also proved to be $C_n$ .

### 4.2.2. Bijection to Standard Young Tableaux

Next, the construction of bijection will be built between binary tree and standard young tableaux, since binary search tree merely fills the corresponding structure with numbers into the nodes.

The bijection is connected through construction by the bridge of correct parentheses sequences because the direct construction between Standard Young tableaux and binary search tree is hard. Since the bijection between Standard Young tableaux and correct parentheses is mentioned before, the ensuing step is about constructing bijection between correct parentheses sequences and binary trees.

Considering the ways of constructing recurrence in the two situations. Let the $P_n$ denotes one permutation of correct parentheses sequence, the correct parentheses sequences split the situation into two parts, one inside the first parentheses and one outside it, that is

$$P_n=(P_k)P_{n-k-1}$$

Or simpler: $(A)B$

And the decomposition of the binary tree is through left subtree and right subtree, and thus a bijection is constructed on

$$\theta: A \to \text{left subtree}, B \to \text{right subtree}$$

As the decomposition finally one subtree of a mere “()”, the tree stops to grow

For example, Figure 3 shows 3 kinds of basic binary tree with their corresponding

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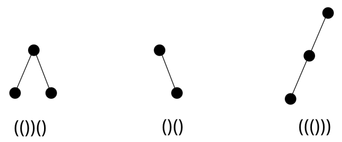


Figure 3. Examples of bijection between binary tree and correct parentheses sequences

Apparently, the figure could reflect that every pair of parentheses corresponds with one node in the binary tree. Let the left parenthesis map to creating the node, and let the right parenthesis map to ticking the node. A direct bijection could be made by the following rules in the Table 1:

Table 1. Rules of direct bijection between binary tree and correct parentheses sequence

| Right: the present sign Down: the previous sign | ( | ) |
|---|---|---|
| none | Create a root node | / |
| ( | Create a left-hand child | Tick the current node |
| ) | Create a right-hand child | Go back to previous ones, until finding one node without a tick and tick it |

Next, by considering the bijection between standard young tableaux and correct parentheses, the rule described in the Table 1 could be transferred into the language of standard young tableaux.

Given a standard young tableau, drawing the ways of ascending sequences, only 4 kinds of motion appear:

1) Horizontally right in the first row
2) Horizontally right in the second row
3) Moves from the first row to the second
4) Moves from the second row to the first

Where they coincidentally correspond to the four main situation in the Table 1, where

1) Horizontally right in the first row <--> "(("
2) Horizontally right in the second row <--> "))"
3) Moves from the first row to the second <--> "()"
4) Moves from the second row to the first <--> ")("

Thus, a bijection between standard young tableaux and binary tree could be created, shown in Table 2.

Table 2. Rules of direct bijection between binary tree and Standard Young Tableaux

| Motion | Horizontally right in the first row | Horizontally right in the second row | Moves from the first row to the second | Moves from the second row to the first |
|---|---|---|---|---|
| Tree creation | Create a left-hand child | Go back to previous ones, until finding one node without a tick and tick it | Tick the current node | Create a right-hand child |

Notice that, because the righter part must be construct as the root is clicked, if a bijection needs to be done by given a binary tree to create a standard young tableaux, the step of creation conversion always follows the rules:

1) Moving nodes always goes through the sequence of root - left - right.
2) Ticking nodes always goes by the sequence of left - root - right.

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### 4.3. Bijection between stack permutation and Standard Young Tableaux

#### 4.3.1. Introduction to stack permutation

A single stack, a data structure that conforms to the last-in, first-out principle, may be used to separate a permutation into sliding order in a stack permutation. In the computer science and the algorithm design, this idea originates from researching how limited operations may be used to redraw data.

Elements can be put onto the Stack in any order, as the order carries for the input sets 1 through n. You may hit the Stack's top element immediately and output the output. A specific permutation is known as Stack sortable if generated with these stack operations.

For instance, the permutation 231 is only stack-sortable, but 312 is not.

The nth Catalan number denotes the number of stack-sortable permutations of length n. Donald Knuth early formed this remarkable connection. The total number of permutations that this particular single-stack sorting procedure may create is expressed in Catalan numbers.

Pattern avoidance may be used to interpret stack-sortable permutations in one of the most elegant ways. There shouldn't be a subsequence of three numbers where it is also the smallest, the second is the largest, and the third is now between. A permutation can also be evaded by minimizing pattern 231.

For instance, 3124 goes stack-sortable because it houses the subsequence 3 1, 1, and 2, while 312 does neither, for example, because that doesn't feature the 231 patterns [10].

#### 4.3.2. Example

Let's take the permutation 312 as an example. We will walk through the stack-sorting process step by step.

Start with the input 1, 2, 3.

Push 1 onto the stack. Then pop it to the output → Output: 1.

Push 2 and then push 3. Pop 3 → Output: 1, 3. Then pop 2 → Output: 1, 3, 2. This gives 132, which is not 312.

Try another strategy: Push 1, then 2, then 3, and then pop 3 → Output: 3. Pop 1 → Output: 3, 1. Pop 2 → Output: 3, 1, 2. Still not 312. So, we must use a different pattern.

To get 312, try this: Push 1, push 2, push 3. Then pop 3 → Output: 3. Pop 1 → Output: 3, 1. Pop 2 → Output: 3, 1, 2. This gives the desired output. Therefore, 312 is stack sortable.

Now consider the permutation 321. Whatever pattern of pushing and popping we choose, we will always encounter a 231 pattern. Hence, 321 is not stack-sortable.

#### 4.3.3. Bijection to Standard Young Tableaux

We utilize a chain of bijections to link stack-sortable permutations to simple Young tableaux (SYT) of shape (n, n). Firstly, stack-sortable permutations are considered to be precisely those that escape the 231 patterns. These permutations coincide bijectivity with Dyck paths, another Catalan object. In turn, Dyck trails correspond bijectively to SYTs of shape (n, n).

Thus, although there is no direct mapping from stack-sortable permutations to SYTs, there is a well-defined pipeline: stack permutation → Dyck path → tableau.

This series of bijections preserves the Catalan count and links sorting operations with representation-theoretic structures. Indexing operations occur in an interconnected manner to

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representation-theoretic structures in this series of bijections, which conserve the Catalan count.

## 4.4. Bijection between noncrossing partition problems and Standard Young Tableaux

### 4.4.1. Introduction to the noncrossing partition

A noncrossing partition is a way of grouping 2n distinct points placed around a circle into n pairs, so that if each pair is joined by a straight line, no two of these lines cross each other inside the circle. That is, the interiors of the segments connecting the paired points do not intersect.

More intuitively, consider 2n points from 1 to 2n labeled in a clockwise order. The relationship could be proved through recursion. In the Figure 4, an arbitrary segment (1, 2k) is chosen (the ending line must be an even number, otherwise the remaining regions' points can't be fully matched).

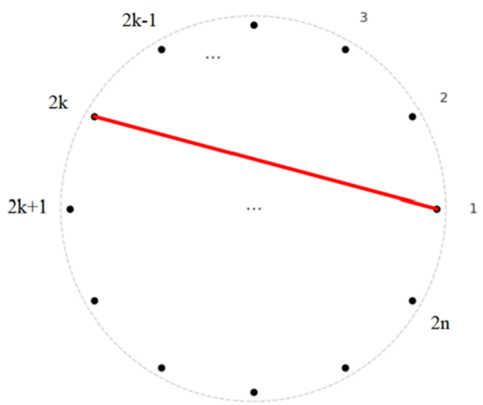


Figure 4. A graph showing one segment (1,2k) dividing the graph

Then the graph is divided into two regions with range [2, 2k-1] and [2k+1, 2n], where the lengths are 2k-2 and 2n-2k, respectively. Moreover, the two regions are independent from each other. Thus, if we denote the ways of noncrossing partition in situation [1,2n] to be $D_n$, with the value of k varies from [1, n], then the recursion equation is derived as

$$D_n = \sum_{k=1}^{n} D_{k-1} D_{n-k},$$

which is identical to the recursive definition of Catalan numbers, it follows that

$$D_n = C_n$$

### 4.4.2. Bijection to Standard Young Tableaux

To construct a bijection, first recall that a bijection is a one-to-one and onto mapping between two sets. In this context, it describes a correspondence between noncrossing partitions of 2n points and standard Young tableaux with two rows. The construction uses recursion based on the first pair, similar to the pattern for correct parentheses sequences. In this correspondence, the smaller number in a pair act like a left parenthesis, and the larger number acts like a right parenthesis. Connecting this to standard Young tableaux, we can give an explicit rule:

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A smaller number in the pair = put into the first row,

A bigger number in the pair = put into the second row.

If the bijection is made from noncrossing partitions to standard Young tableaux, the sequence should be in an order defined by the smaller number (e.x. (1,6) (2,3) (4,5)). And if the bijection is made from standard Young tableaux, the sequence of insertion just follows the sequence in the standard Young tableaux and inserts the corresponding place one by one.

### 4.5. Bijection between covering antichains and Standard Young Tableaux

#### 4.5.1. Introduction to the structure

The Catalan numbers count the number of covering antichains in an interval of length n. This means, the number of ways to cover the interval with antichains with the following properties:

1) All elements are covered

2) None of the antichains is contained in another antichain.

For example, the Figure 5 shows a way of antichains with length 7, from 1 to 8:

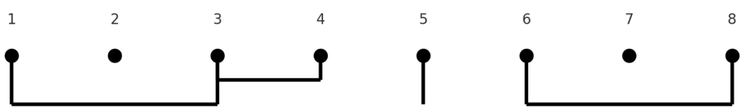


Figure 5. An example of covering antichains with interval [1,8]

The way of proving the number of antichains starts from finding the recursion, just as the method before. Denote the number of antichains of interval for length n is denoted by $D_n$ . Dividing the problem into two minor parts by the first place where the covering antichains are not connected, that is, the first time where the antichains don't share any common subset. For example, the Figure 6 shows the situation of splitting into two minor parts.

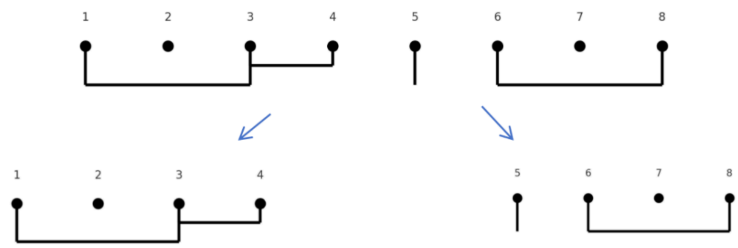


Figure 6. An example of splitting covering antichains with interval [1,8] into two minor parts

If the length of the interval in the first part is k, then clearly the length of the remaining part is n-k. The two parts are independent of each other, but their situations are different. The second part with length n-k will have $D_{n-k}$ ways of covering antichains, since it has no restriction. However, for the first part, it does not simply follow $D_k$ ways of covering antichains: The division follows the first place where the antichains are not connected, which declares that the first part with length of k must be connected overtime. Let's denote the number of ways to manipulate connected covering antichains with length k to be $P_k$ .

Considering one operation, denoted by "Expansion": expand every antichain to include a larger element. For example:

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$$\{1,3\}\{3,4\}\{5\} \rightarrow \{1,4\}\{3,5\}\{5,6\}$$

Actually, it could be seen that every way of covering antichains for an interval of a length n could be operated as a way of connected covering antichains for an interval of a length n+1, because any separation part would coincide due to the extra element added, and no antichains will cover another one totally.

Moreover, this operation could also be reversed; every connected covering antichains could be processed by "Degeneration": Delete the biggest element in every antichain. For example:

$$\{1,4\}\{3,5\}\{4,6\}\rightarrow\{1,3\}\{3,4\}\{4,5\}$$

Notice that this operation could also be applied to any situation of connected covering antichains with length n to get a situation of covering antichains with length n-1.

In all, due to this reversible operation, it follows that every situation of connected covering antichains with length n actually bijects with one situation of covering antichains with length n-1. That is

$$P_k=D_{k-1}.$$

Back to the question, it could be inferred that

$$D_k=\sum_{k=1}^{n} P_k D_{n-k}=\sum_{k=1}^{n} D_{k-1} D_{n-k}.$$

Which is identical to the recursive definition of Catalan numbers, then

$$D_n=C_n$$

### 4.5.2. Bijection to Standard Young Tableaux

Considering the ways of recursion is made. Noticing that the first part of separation, which is connected, actually corresponds to the case of one less length's covering antichains, that is $P_k=D_{k-1}$. Therefore, it could be deduced that one operation of "Expansion" corresponds with adding a new pair of parentheses on the outside. For example:

$\{1\} \rightarrow ()$
$\{1,2\} \rightarrow(())$
$\{1\}\{2\}\rightarrow()()$
$\{1,2\}\{2,3\}\rightarrow(()())$

It could be seen that the number of outer parentheses corresponds with the length of the antichains, since the length of one antichain could decide how many operations it could require. Thus, we could call it "level", for the number of pairs of parentheses the antichain corresponds to.

For an antichain of length n, it will be clear that the level is n+1. For example, $\{1\}$, has a length of 0, its level is then 1, since it only corresponds with a mere pair of parentheses: (), and it could be moreover changed into standard Young tableaux.

It could be seen that the number of outer parentheses corresponds with the length of the antichains, since the length of one antichain could decide how many operations it could require.

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Thus, we could call it "level", for the number of pairs of parentheses the antichain corresponds to.

Therefore, a final bijection could be made: Regarding the length of the antichain to be m and construct the graph to make m more numbers than the second rows. Next, put the next element in the first row, and then the next element in the second row.

For example, if the first antichain is {1,3} then the Standard Young tableaux will be constructed as table 3.

Table 3. Expansion / degeneration operations on covering antichains and their effect on connectivity (schematic examples)

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | | |

If the next antichain is {4}, then the Table will barely be table 4.

Table 4. Bijection recipe from covering antichains of length n to SYT of shape (n,n) (illustrative instance)

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 7 |
| 4 | 5 | 6 | 8 |

(The antichain {4} has a length of 0, and thus the graph must satisfy that the first row has 0 more element than the second row at first. Then a new pair of numbers is added (7 and 8))

When the last antichain is put into operation, if there are still some remaining blanks in the second row, just fill them in an ascending order.

## 5. Conclusion

This paper has explored the bijection between standard Young tableaux of shape (n,n) and 6 classical combinatorics structure. Starting from Correct parentheses sequence, Dyck paths, and binary trees to non-crossing partitions, stack permutations and covering antichains. Each of them embodies the same attributes of recursion in different and surprising way, providing different perspective for understanding Catalan numbers.

Through these bijections, we get a deeper understanding of the Catalan numbers and their many classical structures. They not only serve as an auxiliary tool to help others to comprehend Catalan numbers more profoundly but also allow us to see the essence behind the phenomenon, gaining the core ideas that unify seemingly different structures. Most importantly, these connections reveal a straightforward and elegant correspondence between diverse combinatorial objects, showing the hidden beauty of mathematics.

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