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Entanglement Witnesses Detecting the Bell State

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Abstract. Quantum entanglement is a core resource for quantum information processing, and entanglement witnesses are indispensable tools for detection and verification. This paper first clarifies some already-known properties of entanglement witnesses and the Bell State. By applying mathematical tools such as Kronecker product and matrix algebra, this paper analyzes elements in the Hermitian matrix and discusses the conditions for it to form an entanglement witness. This paper also considers both necessary and sufficient conditions of entanglement witness to impose restrictions on elements and investigated several inequality properties of two-qubit entanglement witness to detect Bell states.

Keywords: Entanglement Witness, Bell State, Hermitian Matrix

1. Introduction

An entanglement witness is a physical notion from quantum mechanics, which has been widely studied in the past decades. First, nonlinear entanglement witnesses have been characterized by matrix algebra [1]. Some universal observables were constructed for detecting two-qubit entanglement based on determinant and separability tests [2]. Further, the optimality of decomposable entanglement witnesses was related to the so-called completely entangled subspaces [3]. They rely on the spanning property, which turns out to be not necessary for some optimal decomposable witnesses [4]. Some authors also investigated the separability of quantum states in terms of a single entanglement witness [5]. For the sake of security, device-independent witnesses of genuine multipartite entanglement have also been proposed [6]. The experimental progress on such witnesses was also reported [7]. On the other hand, the connection between geometric entanglement witnesses and bound entanglement was discovered recently [8]. It was particularly studied for two-qutrit systems with geometry [9]. Further, optimal entanglement witnesses for low-dimensional systems were also detailedly studied [10].

The rest of this paper is organized as follows. In Sec 2, the necessary mathematical background, including matrix operations and the Kronecker product, is introduced. In Sec 3, the notion of entanglement witness (EW) is defined, and some of its fundamental properties, such as the non-negativity of diagonal entries (Lemma 1), are discussed. Section 4 presents the main results of this paper, focusing specifically on the properties of an X-type Hermitian matrix as an EW for Bell states. Five key properties are proposed, applying techniques such as partial derivatives to find necessary inequalities for the matrix elements. Finally, conclusions are drawn in Sec 5.

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2. Mathematics

In this section, the basic knowledge and facts used in this paper are introduced. The notation of a group is (S, +), where the set is represented by S and the operation is +. The first property of a group is closure $\forall x, y \in S, x+y \in S$. The following identities are Associativity, $\forall x, y, z \in S, (x+y)+z = x+(y+z)$, Identity, $\exists e \in S, \forall x \in S, e+x = x$ and Inversion, $\forall x \in S, \exists x^{-1} \in S, x+x^{-1} = e$.

A ring is a set with two operations which satisfy the following restrictions. The notation of a ring is $(S, +, \times)$, where $(S, +)$ is a group that satisfies commutativity, $\forall x, y \in S, x+y = y+x$ (Abel group); $(S, \times)$ is a semigroup that only satisfies the closeness, associativity identity. The $+$ operation satisfies the distribution to the $\times$ operation, $(x+y)z = xz+yz$ namely the right distribution, $z(x+y) = zx+zy$ namely the left distribution. The set $L = \{(x, y) | x, y \in \mathbb{C}\}$ is a linear space when for any $x, y \in L$, and $z \in C$, it holds that $x+y \in L$ and $zx \in L$. If two groups have a bijection and keep the distance of group elements up after the bijection, then the groups are called isomorphic.

For the set S of $m \times n$ matrices, $(S, +)$ is a group. For the matrices $A = [a_{ij}]^{p \times q}$ and $B = [b_{jk}]^{q \times r}$, the following operations are presented. The first operation is multiplication, $AB = C = [c_{ik}]^{p \times r}$ where $c_{ik} = \Sigma_j a_{ij} b_{jk}$. Here each element of A in a row is multiplied by the elements of B in a column corresponding to it. The second operation is Transpose, $A^T = [a_{ji}]^{q \times p}$; By the definitions of the transpose, it can be derived that $(AB)^T = B^T A^T$, furthermore, by mathematics induction, $(A_1 ... A_n)^T = A_n{}^T ... A_1{}^T$.

The Kronecker product is defined as the following form, with $A = [a_{ij}]^{m \times n}$, $B = [b_{jk}]^{p \times q}$,

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \tag{1}$$

The size of $A \otimes B$ is $mp \times nq$. The Kronecker product satisfies the following identities, Associativity. Distribution. For a scalar a , $a \otimes A = A \otimes a = aA$. For conforming matrices $(A \otimes B)(C \otimes D) = AC \otimes BD$. $(A \otimes B)^T = A^T \otimes B^T$. $\mathbf{a}^T \otimes \mathbf{b} = \mathbf{b}\mathbf{a}^T = \mathbf{b} \otimes \mathbf{a}^T$.

3. Entanglement witness

3.1. Conjugate transpose and hermitian matrix

For a matrix $H = [h_{ij}]$, the conjugate transpose of H , H^* , has the property that for each element in $H^* = [\bar{h}_{ij}]$, in other words, $H^* = \bar{H}^T$.

Also, if a matrix H has the property $H^* = H$, then H is a Hermitian Matrix, for example, $H = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$ is a Hermitian Matrix.

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3.2. Density operator and the trace of a matrix

If ρ is an $n \times n$ matrix, and for any $x \in \mathbb{C}^n$, $x^* \rho x \geq 0$, then ρ is a positive semi-definite matrix. The trace of a $n \times n$ matrix W is defined by $Tr(W) = \sum_{i=1}^{n} w_{ii}$, which is the sum of all diagonal elements of a matrix.

3.3. Definition of entanglement witness (EW)

The EW is a matrix in $\mathbb{M}_m \otimes \mathbb{M}_n$, where $\mathbb{M}_m$ and $\mathbb{M}_n$ are respectively the set of $m \times m$ and $n \times n$ matrices. An EW should also satisfy the following identities: the first identity is that if H is an EW, then H is a Hermitian matrix. Next, for any vectors $a \otimes b \in \mathbb{C}^m \otimes \mathbb{C}^n$, it must hold that

$$(a \otimes b)^* H (a \otimes b) \geq 0 \tag{2}$$

There exists a vector $\psi \in \mathbb{C}^m \otimes \mathbb{C}^n$, such that $\psi^* H \psi < 0$. That is, H is not a positive semi-definite matrix. If $Tr(\rho H) < 0$, then the density operator ρ is detected by H.

Now some properties for the EW $H = [h_{ij}] \in \mathbb{M}_2 \otimes \mathbb{M}_2$ are considered.

Lemma 1. It holds that $\mathrm{h_{ii}} \geq 0$ for $\mathrm{i} = 1,2,3,4$.

Proof. Let $a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $a \otimes b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $(a \otimes b)^* = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$, which extracts h_{11} from H by equation (2). By replacing the vectors a and b by orthonormal vectors, h_{11}, h_{22}, h_{33}, h_{44} can be respectively extracted from H. By the second property of EW, it follows that $h_{11}, h_{22}, h_{33}, h_{44} \geq 0$.

Suppose the product matrix $A \otimes B \in \mathbb{GL}(m, \mathbb{C}) \times \mathbb{GL}(n, \mathbb{C})$. Then for an EW $W \in \mathbb{C}^m \otimes \mathbb{C}^n$, one can show that $W_1 = (A \otimes B) W (A \otimes B)^*$ is also an EW.

3.4. Bell state

The Bell state for a two-qubit EW $W \in \mathbb{M}_2 \otimes \mathbb{M}_2$ is determined by a normalized vector $\psi = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ if the matrix W satisfies $Tr(W) = 1$ and $Tr(\psi \psi^* W) < 0$. In addition, $Tr(\psi \psi^* W) = \frac{1}{2}(w_{11} + w_{14} + w_{41} + w_{44})$, therefore, the condition is equivalent to $w_{11} + w_{14} + w_{41} + w_{44} < 0$.

4. Result

Consider an EW

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$$W = \begin{bmatrix} w_{11} & 0 & 0 & w_{14} \\ 0 & w_{22} & w_{23} & 0 \\ 0 & w_{32} & w_{33} & 0 \\ w_{41} & 0 & 0 & w_{44} \end{bmatrix} \tag{3}$$

4.1. Property 1

For any elements w_{ii} in equation (3), it holds that $w_{ii} \geq 0$ and $\Sigma w_{ii} = 1$.

Proof. Since an EW must be a Hermitian matrix, it holds that $W = W^*$, which gives $w_{11}, w_{22}, w_{33}, w_{44} \in \mathbb{R}$. By Lemma 1, it can be further concluded that $w_{11}, w_{22}, w_{33}, w_{44} \geq 0$ and $w_{41} = \overset{*}{w}_{14}, w_{32} = \overset{*}{w}_{23}$.

Furthermore, if W is considered in bell state, then $Tr(W) = w_{11} + w_{22} + w_{33} + w_{44} = 1$, $w_{11} + w_{14} + w_{41} + w_{44} < 0$.

4.2. Property 2

For elements $w_{23}, w_{32}, w_{41}, w_{14}$ in equation (3): $w_{23} = w_{32} \geq 0$, $w_{14} = w_{41} \geq 0$

Proof. If two matrices $\mathrm{a} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}$ and $\mathrm{b} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{bmatrix}$ are considered. By the conclusion that

$$W^{'} = \left(\mathrm{a} \otimes \mathrm{b}\right)^{*} W \left(\mathrm{a} \otimes \mathrm{b}\right)$$

$$= \begin{bmatrix} w_{14} & 0 & 0 & w_{14}e^{-i(\alpha+\theta)} \\ 0 & w_{22} & w_{23}e^{i(\alpha-\theta)} & 0 \\ 0 & w_{23}e^{-i(\alpha-\theta)} & w_{33} & 0 \\ w_{41}e^{i(\alpha+\theta)} & 0 & 0 & w_{44} \end{bmatrix} \tag{4}$$

is an EW, with the property of $W = W^*$ and appropriate $\alpha, \theta, w_{23} = w_{32} \geq 0, w_{14} = w_{41} \geq 0$.

In this case, any EW that has the same structure as equation (3), can be simplified to an EW $W^{'}$, in which all the elements in the matrix are in $\mathbb{R}$. Therefore, in the following discussion of W, W is considered as a real matrix.

4.3. Property 3

The elements in the leading diagonal and the minor diagonal of equation (3) satisfy,

$$w_{11} + w_{22} + w_{33} + w_{44} \geq w_{41} + w_{14} + w_{32} + w_{23} \tag{5}$$

Proof. Let two vectors a, b be $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, respectively, where the elements are complex numbers. If any of a_1, a_2, b_1, b_2 equals to zero,

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$$(\mathrm{a}\otimes\mathrm{b})^{*}W(\mathrm{a}\otimes\mathrm{b})\geq 0 \tag{6}$$

Therefore, a, b can be simplified to the form $\begin{bmatrix} z_1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} z_2 \\ 1 \end{bmatrix}$, respectively. As a result,

$$(\mathrm{a}\otimes\mathrm{b})^{*}W(\mathrm{a}\otimes\mathrm{b})=$$

$$w_{11}|z_1z_2|^2+w_{41}z_1z_2+w_{22}|z_1|^2+w_{32}z_1\bar{z_2}$$

$$+w_{23}\bar{z_1}z_2+w_{33}|z_2|^2+w_{14}z_1\bar{z}_2+w_{44} \tag{7}$$

Since $w_{14}=w_{41}$ and $w_{23}=w_{32}$, this does not give further restrictions, as a result, when considering the minimum of $\mathrm{a}\otimes\mathrm{b})^{*}W(\mathrm{a}\otimes\mathrm{b})$, z_1, z_2 could be represented by two real numbers x, y. Therefore, a, b can be set as $\begin{bmatrix} x \\ 1 \end{bmatrix}$ and $\begin{bmatrix} y \\ 1 \end{bmatrix}$, respectively. In this way,

$$(\mathrm{a}\otimes\mathrm{b})^{*}W(\mathrm{a}\otimes\mathrm{b})=$$

$$xy(xyw_{11}+w_{41})+x(xw_{22}+yw_{32})+y(xw_{23}+yw_{33})+xyw_{14}+w_{44}\geq 0 \tag{8}$$

Let $x=1, y=-1$,

$$w_{11}-w_{41}+w_{22}-w_{32}-w_{23}+w_{33}-w_{14}+w_{44}\geq 0$$

$$\Rightarrow w_{11}+w_{22}+w_{33}+w_{44}\geq w_{41}+w_{14}+w_{32}+w_{23} \tag{9}$$

In this case, W can be re-written in the form:

$$W=\begin{bmatrix} w_{11} & 0 & 0 & b \\ 0 & w_{22} & a & 0 \\ 0 & a & w_{33} & 0 \\ b & 0 & 0 & 1-w_{11}-w_{22}-w_{33} \end{bmatrix} \tag{10}$$

4.4. Property 4

For elements a, b and diagonal elements w_{ii}, the following inequality holds:

$$\left(b^2-w_{11}\left(1-w_{11}-w_{22}-w_{33}\right)\right)\left(a^2-w_{22}w_{33}\right)<0 \tag{11}$$

Proof. To satisfy the condition that equation (10) is not a semi-definite matrix, it must hold that

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$$det\begin{bmatrix} w_{11} & b \\ b & 1 - w_{11} - w_{22} - w_{33} \end{bmatrix} < 0 \tag{12}$$

or

$$det\begin{bmatrix} w_{22} & a \\ a & w_{33} \end{bmatrix} < 0 \tag{13}$$

Only one of them would be satisfied.

4.5. Property 5

Let $s = w_{14} + w_{41} + w_{23} + w_{32}$, the leading diagonal elements $w_{11}, w_{22}, w_{33}, w_{44}$ in equation (3) satisfy

$$w_{44} \geq \frac{w_{33}(sk - 2w_{22})^2}{4w_{11}w_{22}} \tag{14}$$

where $k = \sqrt{\frac{w_{22}}{w_{33}}}$.

Proof. Consider

$$f(x,y) = w_{11}(xy)^2 + (w_{41} + w_{14} + w_{32} + w_{23})xy + w_{22}x^2 + w_{33}y^2 + w_{44} \tag{15}$$

It has the extrema at $\nabla f(x,y) = \frac{\partial f}{\partial x}\widehat{x} + \frac{\partial f}{\partial y}\widehat{y} = 0$ for any vector $\widehat{x}, \widehat{y}$ .Let $A = \frac{\partial^2 f}{\partial x}$, $B = \frac{\partial^2 f}{\partial x \partial y}$, $C = \frac{\partial^2 f}{\partial y}$, then

$$AC - B^2 =$$

$$-12w_{11}^2x^2y^2 + 8w_{11}(w_{14} + w_{41} + w_{23} + w_{32})xy$$

$$+4w_{11}w_{22}x^2 + 4w_{11}w_{33}y^2 + 4w_{22}w_{33} + (w_{14} + w_{41} + w_{23} + w_{32})^2 \tag{16}$$

By the conclusion that, if point (x_0, y_0) satisfies that $\frac{\partial f(x_0,y_0)}{\partial x} = \frac{\partial f(x_0,y_0)}{\partial y} = 0$ and $AC - B^2 > 0$ while $A > 0$ (It is evident to show that $A > 0$ for any (x,y)), then (x_0, y_0) is the minimum of $f(x,y)$ .Evaluate $\frac{\partial f(x_0,y_0)}{\partial x} = \frac{\partial f(x_0,y_0)}{\partial y} = 0$ gives that

$$\begin{cases} 2w_{11}y^2x + (w_{14} + w_{41} + w_{23} + w_{32})y + 2w_{22}x = 0 \\ 2w_{11}x^2y + (w_{14} + w_{41} + w_{23} + w_{32})x + 2w_{33}y = 0 \end{cases} \tag{17}$$

Define the parameter $s = w_{14} + w_{41} + w_{23} + w_{32}$ to simplify notation. The system becomes:

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$$\begin{cases} 2w_{11}xy^2 + sy + 2w_{22}x = 0 \\ 2w_{11}x^2y + sx + 2w_{33}y = 0 \end{cases} \tag{18}$$

This system is solved under the standard assumptions $w_{22} > 0$ and $w_{33} > 0$. Define $k = \sqrt{\frac{w_{22}}{w_{33}}}$.

The critical points are given by:

$$(\pm\sqrt{-\frac{sk+2w_{22}}{2w_{11}k^2}},\ \pm k\sqrt{-\frac{sk+2w_{22}}{2w_{11}k^2}}), for\ -\frac{sk+2w_{22}}{2w_{11}k^2} > 0$$

$$(\pm\sqrt{\frac{sk-2w_{22}}{2w_{11}k^2}},\ \mp k\sqrt{\frac{sk-2w_{22}}{2w_{11}k^2}}), for\ \frac{sk-2w_{22}}{2w_{11}k^2} > 0$$

At these points:

$$f\left(x, y\right) = w_{44} - \frac{w_{33}(sk-2w_{22})^2}{4w_{11}w_{22}} \tag{19}$$

Non-negativity requires:

$$w_{44} \geq \frac{w_{33}(sk-2w_{22})^2}{4w_{11}w_{22}} \tag{20}$$

In addition, the trivial solution $(0,0)$ gives $f(0,0) = w_{44} \geq 0$, which is already satisfied by previous properties. When $w_{11} = 0$, $f(x,y)$ reduces to a quadratic form, and non-negativity requires $w_{22}w_{33} \geq s^2/4$.

The critical point analysis guarantees that $f(x,y)$ has no negative minima, ensuring non-negativity on all product states.

However, the equation

$$w_{44} \geq \frac{w_{33}(sk-2w_{22})^2}{4w_{11}w_{22}} \tag{21}$$

is not a sufficient condition for equation (3) to be an entanglement witness. The inequality is satisfied while taking all $w_{i(5-i)} = w_{ii} = 1$, while this cannot satisfy Property 1, $\Sigma w_{ii} = 1$ .However, for equation (3) which satisfies all 5 properties, it is sufficient to be an entanglement witness. Property 5 ensures that for all $\mathrm{a} = \begin{bmatrix} x \\ 1 \end{bmatrix}$ and $\mathrm{b} = \begin{bmatrix} y \\ 1 \end{bmatrix}$,

$$(\mathrm{a} \otimes \mathrm{b})^* W (\mathrm{a} \otimes \mathrm{b}) \geq 0 \tag{22}$$

is always satisfied. On the other hand, property 1-4 restricts that equation (3) satisfies other definitions of an entanglement witness. In other words, satisfying property 1 to 5 is the sufficient condition for entanglement witness. This leads to property 6.

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4.6. Property 6

If a 2×2 matrix W satisfy all properties from 1 to 5 and has the same elements as equation (3) then W is an entanglement witness.

Proof. If property 1, property 2 and property 3 are satisfied, then:For any elements w_{ii} in W, it holds that $w_{ii} \geq 0$ and $\Sigma w_{ii} = 1$.

For elements $w_{23}, w_{32}, w_{41}, w_{14}$ in W:

$$w_{23} = w_{32} \geq 0$$

$$w_{14} = w_{41} \geq 0$$

The elements in the leading diagonal and the minor diagonal of W satisfy,

$$w_{11} + w_{22} + w_{33} + w_{44} \geq w_{41} + w_{14} + w_{32} + w_{23} \tag{23}$$

Hence, W is a Hermitian matrix and satisfies $Tr(W) = 1$ Next, the following lemma is presented:

Lemma 2. If property 4 is satisfied, then there exists a vector $\psi \in \mathbb{C}^m \otimes \mathbb{C}^n$, such that $\psi^* H \psi < 0$.

Proof. Suppose W is a semi-definite matrix, $(b^2 - w_{11}(1 - w_{11} - w_{22} - w_{33}))(a^2 - w_{22}w_{33}) < 0$ contradicts with the definition of semi-definite matrix that $\det \begin{bmatrix} w_{11} & b \\ b & 1 - w_{11} - w_{22} - w_{33} \end{bmatrix} < 0$ and $\det \begin{bmatrix} w_{22} & a \\ a & w_{33} \end{bmatrix} < 0$ must be positive.

Lemma 3. If property 5 is satisfied, then for all $a = \begin{bmatrix} x \\ 1 \end{bmatrix}$ and $b = \begin{bmatrix} y \\ 1 \end{bmatrix}$, $(a \otimes b)^* W (a \otimes b) \geq 0$ is always satisfied.

Proof. From the proof of Property 2 and 3, a and b can be restricted in $\mathbb{R} \times \mathbb{R}$. On the other hand, if Property 5 is guaranteed, since the minimum value of equation (15) is equal to $w_{44} - \frac{w_{33}(sk - 2w_{22})^2}{4w_{11}w_{22}}$ and $w_{44} \geq \frac{w_{33}(sk - 2w_{22})^2}{4w_{11}w_{22}}$, equation (15) is ensured to be positive for all a and b in $\mathbb{C} \times \mathbb{C}$.

Thus, W satisfies all the conditions to become an entanglement witness.

5. Conclusions

The entanglement witnesses detecting the Bell states have been studied from a viewpoint of linear algebra and discussed several properties of entanglement witnesses. In particular, it has been shown that the matrix entries of the witnesses are subject to some equations, which determine the structure of witness. Furthermore, several criteria were developed for the matrix to be an entanglement witness. An open problem arising from this paper is to totally construct all two-qubit entanglement witnesses detecting two-qubit Bell states, which can be further extended to high-dimensional Bell

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states. Another interesting issue is to extend the Bell state to three-qubit GHZ state, whose genuine entanglement may be detected by some bipartite entanglement witnesses, too.

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