

# *Catalan Numbers: Generalization from Fuss Catalan Numbers to Related Applications*

**Pengxiao Xiang**

*Ulink College Guangzhou, Guangzhou, China*

*Corresponding Author. Email: penxiang2650@guiscn.com*

**Abstract.** *Catalan numbers* are a concept derived from the study of *convex polygon dissections*. Based on this idea, this article aims to discover the importance and similarity of Catalan numbers in solving different problems in enumerative combinatorics, according to the generalization of its recursive formula. With the introduction of *Fuss Catalan numbers*, which is a generalization of Catalan numbers, the following section will show its relation with *j-ary trees* and discover solutions of specific problems from different perspective. The article also expands Catalan numbers in particular *Avoiding Permutations* and *Standard Young Tableaux*, which indicates the breadth of applications of Catalan numbers.

**Keywords:** Catalan numbers, convex polygon dissections, Fuss Catalan numbers, j-ary trees, Avoiding Permutations, Standard Young Tableaux

## 1. Introduction

Catalan numbers (which denote  $C_n$ ), named after the Belgian mathematician Eugène Charles Catalan, are a sequence of numbers with general formula  $C_n = \frac{1}{n+1} \binom{2n}{n}$ . Similarly, its generalization, the Fuss Catalan numbers (which denote  $C_{n,k}$ ) also have a general formula  $C_{n,k} = \frac{1}{kn+1} \binom{(k+1)n}{n}$ .

The 3-step proof of both of the concepts will be shown as follows, with the recursive formula, the generating function and the general formula being obtained respectively[1]. This proof provide a solid evidence for the way of mapping which can construct a connection between Catalan numbers and realistic problems. As the recursive formula being applied, complex problems can be simplified immediately through such mapping. A range of examples will be shown around it.

## 2. Review of catalan numbers

### 2.1. Definition

Catalan numbers can be defined in a range of ways. For instance, the linear sequence of Last In First Out (LIFO), which is also called “stack”. This article will define it from the perspective of convex polygon dissections, which is a classic problem raised by Euler in 1751: Notice that a  $(n + 2)$ -convex polygon in the plane can be denoted as  $P_{n+2}$ . Then  $C_n$ , which is the  $n^{th}$  Catalan number can be used to represent the number of ways to triangulate the convex polygon  $P_{n+2}$ . An example of  $C_4 = 14$  is shown (Fig.1).

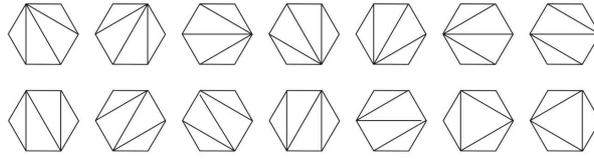


Figure 1:  $C_4 = 14$

## 2.2. Proof for the general formula

Catalan numbers have a general formula (where  $n$  is an integer):

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

The path to a general formula for the Catalan numbers originates from establishing a recurrence relation[1]. Consider a convex polygon with  $n + 2$  vertices, and let  $f(n)$  represent its total number of triangulations. To derive a recurrence for  $f(n)$ , we can fix a specific edge, say between vertices  $A_1$  and  $A_{n+2}$ , as the base of a triangle. The third vertex of this triangle must be one of the interior vertices  $A_k$  for  $k = 2, 3, \dots, n + 1$ .

This initial triangle subdivides the original polygon into three distinct parts: the triangle itself, a smaller polygon with  $k$  vertices (from  $A_1$  to  $A_k$ ), and another polygon with  $n - k + 3$  vertices (from  $A_k$  to  $A_{n+2}$ ). The number of ways to triangulate the entire structure is the product of the ways to triangulate these two independent sub-polygons. A polygon with  $m$  vertices has  $f(m - 2)$  triangulations, and we note the base case  $f(1) = 1$  (a single triangle has one trivial triangulation). Summing over all possible positions for the vertex  $A_k$  yields the recurrence [2]:

$$f(n) = \sum_{k=2}^{n+1} f(k-2) \cdot f(n-k+1)$$

This summation can be clarified by a change of index. Letting  $i = k - 2$ , the limits change to  $i = 0$  to  $i = n - 1$ , and the term  $f(n - k + 1)$  becomes  $f(n - i - 1)$ . This transforms the recurrence into its more standard form:

$$f(n) = \sum_{i=0}^{n-1} f(i) \cdot f(n-i-1)$$

Since the Catalan number  $C_n$  is defined as  $f(n)$ , by substitute  $n$  with  $n + 1$ , we immediately obtain the fundamental recurrence relation defining the sequence:

$$C_{n+1} = \sum_{i=0}^n C_i C_{n-i}$$

(Notice that  $n \geq 0$  and  $C_0 = 1$ )

Progressing from this recurrence to an explicit closed-form expression is efficiently achieved using **generating functions** [2]. We define the ordinary generating function for the sequence of Catalan numbers as:

$$C(x) = \sum_{n=0}^{\infty} C_n x^n = 1 + x + 2x^2 + 5x^3 + \dots$$

Considering the square of this generating function:

$$\begin{aligned} C(x)^2 &= (C_0 + C_1x + C_2x^2 + \dots)(C_0 + C_1x + C_2x^2 + \dots) \\ &= \sum_{n=0}^{\infty} \left( \sum_{i=0}^n C_i C_{n-i} \right) x^n \end{aligned}$$

According to the **recursive formula**, substituting the coefficients  $\sum_{i=0}^n C_i C_{n-i}$  with the identical value  $C_{n+1}$

$$C(x)^2 = \sum_{n=0}^{\infty} C_{n+1} x^n = \frac{1}{x} \sum_{n=1}^{\infty} C_n x^n = \frac{C(x) - C_0}{x} = \frac{C(x) - 1}{x}$$

Solve the quadratic equation in  $C(x)$ . As  $C_0 = 1$ , it requires  $\lim_{x \rightarrow 0} C(x) = 1$ :

$$C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$$

Applying **Binomial theorem** to expand  $\sqrt{1 - 4x}$

$$C(x) = \frac{1}{2x} \left( 1 - \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-4x)^n \right) = -\frac{1}{2} \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n+1} (-4)^{n+1} x^n$$

which indicates the coefficients

$$C_n = -\frac{1}{2} \binom{\frac{1}{2}}{n+1} (-4)^{n+1}$$

Notice that  $n$  is an integer, and applying the Binomial theorem to  $\binom{\frac{1}{2}}{n+1}$  as well [2]. The equation can be simplified into:

$$C_n = \frac{(2n)!}{n!(n+1)!} = \frac{1}{n+1} \binom{2n}{n}$$

### 3. Fuss catalan numbers

#### 3.1. Definition

For integers  $k \geq 1$  and  $n \geq 0$ ,  $C_{n,k}$  are given by the closed-form expression [3]:

$$C_{n,k} = \frac{1}{kn+1} \binom{(k+1)n}{n}$$

#### 3.2. Proof (three steps)

##### 3.2.1. The recursive formula

For a  $(kn+2)$ -gon, label the vertices clockwise with letters  $V_1, V_2, \dots, V_{kn+2}$ . Then dissect the  $(kn+2)$ -gon and fix the vertices  $V_1, V_{kn+2}$  as one side of the  $(k+2)$ -gon. Similarly, label the vertices of the  $(k+2)$ -gon clockwise with the letters  $V_{a_1}, V_{a_2}, \dots, V_{a_{k+1}}, V_{a_{k+2}}$ . An equation forms as follows:

$$a_1 = 1, a_{k+2} = kn + 2 \tag{1}$$

Assume that the polygon with vertices  $V_{a_j}, V_{a_{j+1}}, \dots, V_{a_{j+1}-1}, V_{a_{j+1}}$  can be divided into  $(k + 2)$ -gons in  $C_{i_j, k}$  ways (Notice that  $j \geq 1$ ). It contains  $a_{j+1} - (a_j - 1)$  vertices which will form a  $(i_j k + 2)$ -gon [3]. Equations form as follows (where  $j$  and  $i_j$  are both integers):

$$a_{j+1} - (a_j - 1) = i_j k + 2$$

$$i_j = \frac{a_{j+1} - a_j - 1}{k} \tag{2}$$

Substituting with Equation (2) and Equation (1), respectively

$$\sum_{j=1}^{k+1} i_j = \sum_{j=1}^{k+1} \frac{a_{j+1} - a_j - 1}{k} = \frac{(a_{k+2} - a_1) - (k + 1)}{k} = \frac{(kn + 2 - 1) - (k + 1)}{k} = n - 1$$

Thus

$$C_{n, k} = \sum_{i_1 + i_2 + \dots + i_{k+1} = n-1} C_{i_1, k} \cdot C_{i_2, k} \cdots C_{i_{k+1}, k}, \quad n \geq 1$$

For  $n = 1, C_{1, k} = 1$

$$C_{1, k} = \sum_{i_1 + i_2 + \dots + i_{k+1} = n-1} C_{i_1, k} \cdot C_{i_2, k} \cdots C_{i_{k+1}, k} = C_{0, k}^k$$

As  $C_{0, k} = 1$ , the recursive formula is well defined for  $n \geq 0$

### 3.2.2. Generating function

Denote  $C_k(x) = \sum_{n=0}^{\infty} C_{n, k} \cdot x^n = C_{0, k} + C_{1, k}x + C_{2, k}x^2 + \dots$

Expanding the brackets and substituting the coefficients of  $x^n (n \geq 0)$  with the **recursive formula**

$$C_k(x)^{k+1} = C_{0, k}^{k+1} + (k + 1)C_{0, k}^k C_{1, k}x + \dots = C_{1, k} + C_{2, k}x + \dots = \sum_{n=0}^{\infty} C_{n+1, k} \cdot x^n$$

Then

$$1 + xC_k(x)^{k+1} = 1 + \sum_{n=0}^{\infty} C_{n+1, k} \cdot x^{n+1} = C_{0, k}x^0 + \sum_{n=1}^{\infty} C_{n, k} \cdot x^n = \sum_{n=0}^{\infty} C_{n, k} \cdot x^n = C_k(x)$$

So

$$C_k(x) = 1 + xC_k(x)^{k+1}$$

### 3.2.3. Lagrange inversion

According the the literature *Lagrange Inversion: When and How*, considering a such transformation [4]:

$$[x^n]\phi(f(x)) = \frac{1}{n}[t^{n-1}](\phi'(t)R(t)^n), n \in \mathbb{Z}, n \neq 0$$

Notice  $f(x) = xR(f(x))$ . Then take  $f(x) = C_k(x) - 1, \phi(t) = t$ .  
Substituting  $C_k(x)$  with the generating function [5]

$$f(x) = x(1 + f(x))^{k+1}, R(t) = (1 + t)^{k+1}$$

Now, the equation becomes

$$[x^n]f(x) = \frac{[t^{n-1}](1 + t)^{n(k+1)}}{n}$$

As regards the equation above, according to the binomial expansion [5]

$$RHS = \frac{1}{n} \binom{n(k+1)}{n-1}$$

$$LHS = [x^n]f(x) = [x^n]C_k(x) = C_{n,k}$$

So, for  $n \geq 1, LHS = RHS$

$$C_{n,k} = \frac{1}{n} \binom{n(k+1)}{n-1} = \frac{[n(k+1)]!}{n(n-1)!(k+1)!} = \frac{1}{kn+1} \frac{[n(k+1)]!}{n!(kn)!} = \frac{1}{kn+1} \binom{n(k+1)}{n}$$

(As  $C_{0,k} = 1$ , the general formula is well defined for  $n \geq 0$ )

## 4. Generalization in specific questions

### 4.1. $j$ -ary trees

**The Fuss-Catalan number  $C_{n,k}$  naturally enumerates the number of full  $(k+1)$ -ary trees that contain exactly  $n$  internal nodes [6].**

This combinatorial interpretation arises from the recursive structure inherent to such trees. The proof begins by designating one node as the root, leaving  $n-1$  internal nodes to be distributed among the  $k+1$  subtrees originating from the root. Let  $i_j$  denote the number of nodes in the  $j$ -th subtree. These values must satisfy the constraint:

$$\sum_{j=1}^{k+1} i_j = n - 1$$

Since  $i_j < n$ , and  $C_{i_j,k}$  represents the amounts of subtrees that would have  $i_j$  vertices

$$C_{n,k} = \sum_{i_1+i_2+\dots+i_{k+1}=n-1} C_{i_1,k} \cdot C_{i_2,k} \cdot \dots \cdot C_{i_{k+1},k}$$

Conforms to the previous recursive formula.

### 4.2. Sequences

**$C_{n,k}$  represents the number of sequences of  $n$   $k$ 's and  $kn(-1)$ 's which conforming every partial sum is nonnegative.**

(Proof is shown as follows by constructing a bijection between trees and sequences)

### 4.2.1. From sequences to trees

For any sequence of  $n$   $k$ 's and  $kn$   $(-1)$ 's, it satisfies that every partial sum would be nonnegative. Construct a perfect  $(k + 1)$ -ary tree. Then, define the following operations:

1.  $k$ : Mark the node and move to its first child node
2.  $-1$ : For the  $i^{th}$  child node, if
  - $1 \leq i \leq k$ : Move to the child node  $(i + 1)$
  - $i = k + 1$ : Move back to the node next to its parent node
 Repeat this operation until the node next to its parent node is not the  $i$  th node of its level

Perform the operations in order of the sequence, a graph will be obtained. **To prove the graph obtained is a  $(k + 1)$ -ary tree which contains  $n$  nodes:**

#### Step 1 (Prove that there are $n$ nodes on the perfect $(k + 1)$ -ary tree)

1.  $n$  nodes.  
Each operation of  $k$  will connect the node to its parent node. Also, with respect to the first operation, the root node does not have its parent node. Thus, there are  $(n - 1)$  edges.
2. On the perfect  $(k + 1)$ -ary tree.  
Once at the  $(k + 1)^{th}$  node, the next operation will move it to its first child node or the node next to its parent node. Thus, the only situation to leave this perfect  $(k + 1)$ -ary tree is moving to the node next to the root node. Operation  $k$  will move it to the next level and  $k$  of operation  $(-1)$ 's will move it back to the previous level. Meanwhile, every partial sum is nonnegative. Thus, it will never move to the level before the level of the root node, leading to the movement to the level of the root node twice (One is at the root node, before the first operation and the other one is at the node next to the root node, after the last operation, which will not be marked). So, every marked node is on the perfect  $(k + 1)$ -ary tree.

#### Step 2 (Prove that this graph is a tree):

1. Connectivity.  
If the node is marked, then its parent node will also be marked. Thus, it is connected to the root node. So, any of the two nodes is connected if they are connected to the root node separately.
2. Acyclicity.  
operations are performed on a perfect  $(k + 1)$ -ary tree, which does not have a cycle. So, this graph has no cycle.

#### Step 3 (Prove that it is a $(k + 1)$ -ary tree at most)

As previous proof, once at the  $(k + 1)^{th}$  node, the next operation will move it to its first child node or the node next to its parent node. It is impossible to move to the  $(k + 2)^{th}$  node. So, this tree have at most  $(k + 1)$ -ary, it is a  $(k + 1)$ -ary tree.

### 4.2.2. From trees to sequences

For any  $(k + 1)$ -ary tree with  $n$  vertices, define the following operations to convert the tree into sequences. Start from the root node. The first term in the sequences is  $k$ . Then, move to the first child node.

1. If the node is not empty: Add  $k$  by the end of the series and move to the first child node of this node

2. If the node is empty
  - For the non- $(k + 1)^{th}$  node: Move to the next node in its same level, adding  $(-1)$
  - For the  $(k + 1)^{th}$  node: Move to the node next to its parent node. Repeat this operation until the node next to its parent node is not the  $i$  th node of its level. Then adding  $(-1)$
3. If the node is not on the  $(k + 1)$ -ary tree: The sequence ends.

**Prove the sequence obtained has  $n$  of  $k$ 's and  $kn$  of  $(-1)$ 's and its partial sum is always non-negative:**

**Step 1 (Prove that all partial sum of the sequence is nonnegative)**

This operation can be seen as the inverse operation of the operation from sequence to tree. Thus, similarly, "k" will move to the next level; k of "-1" will move back to the previous level. If the partial sum is negative, then the tested node will at the level before the level of root node. So, the whole partial sum of the sequence is nonnegative.

**Step 2 (Prove for  $n$  of  $k$ 's)**

As the  $(k + 1)$ -ary tree has  $n$  nodes, once the nodes of that tree have been tested, adding a  $k$ . So, the series contains  $n$  of  $k$ 's

**Step 3 (Prove for  $kn$  of  $(-1)$ 's)**

- If there are more  $(-1)$ 's than  $kn$ : As the proof shown in Step 1, it will move to the outsides
- If there are less  $(-1)$ 's than  $kn$ : The tree has not been tested completely, which means the sequence is not complete

So, there must be  $kn$  of  $(-1)$ 's.

**4.3. Bracketings**

$C_{n,k}$  may represent the number of bracketings in a string of  $(kn + 1)$  letters subject to a nonassociative  $(k + 1)$ -ary operation.

Assume a  $C_{i,k}$  describing the number of bracketings in a string of  $(ki + 1)$  letters subject to a nonassociative  $(k + 1)$ -ary operation has been proved. Regard a letter or bracket as one element. Thus, there will be  $(k + 1)$  elements in the outermost layer.

Assume that the  $j^{th}$  element contains  $(ki_j + 1)$  letters

$$\sum_{j=1}^{k+1} ki_j + 1 = k \sum_{j=1}^{k+1} i_j + k + 1 = kn + 1$$

$$\sum_{j=1}^{k+1} i_j = n - 1$$

As  $C_{i_j,k}$  represents the number of permutation for each element

$$C_{n,k} = \sum_{i_1+i_2+\dots+i_{k+1}=n-1} C_{i_1,k} \cdot C_{i_2,k} \cdot \dots \cdot C_{i_{k+1},k}$$

Conforms to the previous recursive formula.

## 5. Proof that catalan numbers count 132-Avoiding permutations

### 5.1. Definitions

A permutation  $\sigma = [\sigma(1), \sigma(2), \dots, \sigma(n)]$  **avoids the pattern 132** if there are no indices  $i < j < k$  [7] such that:

$$\sigma(i) < \sigma(k) < \sigma(j)$$

**Example:**

- $\sigma = [3, 4, 5, 2, 1]$  avoids 132
- $\sigma = [5, 1, 3, 4, 2]$  contains 132

### 5.2. Combinatorial structure

To analyze where the largest element  $n$  would locate in all these permutations.

#### 5.2.1. Position of $n$

For a 132-avoiding permutation denoted as  $\sigma$  in which  $\sigma(k) = n$ :

- The left segment:  $\sigma(1), \dots, \sigma(k-1)$
- The right segment:  $\sigma(k+1), \dots, \sigma(n)$

#### 5.2.2. Critical condition

It is obvious that: **Every element in the left segment must be greater than every element in the right segment**[7].

#### 5.2.3. Consequences

1. Left segment:  $\{n-1, n-2, \dots, n-k+1\}$  which is a 132-avoiding permutation ( $n-1$  is the second largest number after  $n$ , and there are  $k-1$  elements in the left segment.)
2. Right segment:  $\{1, 2, \dots, n-k\}$  which is a 132-avoiding permutation as well ( $n-k$  is the largest number which is smaller than  $n-k+1$ .  $n-k$  elements are contained in the right segment.)

For a fixed  $k$ , there are  $a_{k-1} \cdot a_{n-k}$  permutations.

### 5.3. Recurrence relation

Summing over all positions of  $n$  [8]:

$$a_n = \sum_{k=1}^n a_{k-1} a_{n-k}, \quad a_0 = 1$$

### 5.4. Equivalence to catalan recurrence

The Catalan recurrence is:

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1} = \sum_{k=1}^n C_{k-1} C_{n-k}$$

Thus  $a_n$  satisfies the same recurrence as  $C_n$  [9].

## 6. Standard Young Tableaux and Catalan Numbers

### 6.1. Definitions

- **Standard Young Tableau (SYT):** A Standard Young Tableau is a Young tableau in which the boxes are filled with distinct integers from 1 to  $n$  (where  $n$  is the number of boxes) [10], satisfying:
  - Row strictness: Entries increase strictly left to right in every row.
  - Column strictness: Entries increase strictly top to bottom in every column.
- **Hook-Length Formula:** The hook-length formula states that the number of SYT of shape  $\lambda$ , denoted  $f^\lambda$ , is given by:

$$f^\lambda = \frac{n!}{\prod_{(i,j) \in \lambda} h(i,j)}$$

In this equation,  $n = |\lambda|$  (total cells).

### 6.2. Proof for Shape $(n, n)$

Consider the rectangular partition  $\lambda = (n, n)$  with two rows of equal length of  $n$  included. In this condition,  $2n$  cells are contained in the Young diagram.

#### Step 1: Compute Hook-Lengths

For each cell  $(i, j)$  in  $\lambda$ :

- **Row 1 ( $i = 1$ ):**

$$h(1, j) = \underbrace{(n - j)}_{\text{right}} + \underbrace{1}_{\text{below}} + \underbrace{1}_{\text{the cell (1, j) itself}} = n - j + 2$$

- **Row 2 ( $i = 2$ ):**

$$h(2, j) = \underbrace{(n - j)}_{\text{right}} + \underbrace{0}_{\text{below}} + \underbrace{1}_{\text{the cell (2, j) itself}} = n - j + 1$$

#### Step 2: Find the Product of Hook-Lengths

The product over all of the cells is:

$$\prod_{(i,j) \in \lambda} h(i, j) = \underbrace{\prod_{j=1}^n h(1, j)}_{\text{row 1}} \times \underbrace{\prod_{j=1}^n h(2, j)}_{\text{row 2}} = \prod_{j=1}^n (n - j + 2) \times \prod_{j=1}^n (n - j + 1)$$

Reindex with  $k = n - j + 1$  (so  $j = 1 \Rightarrow k = n, j = n \Rightarrow k = 1$ ):

$$\prod_{j=1}^n (n - j + 2) = \prod_{k=1}^n (k + 1) = 2 \cdot 3 \cdot \dots \cdot (n + 1) = \frac{(n + 1)!}{1} = (n + 1)!$$

$$\prod_{j=1}^n (n - j + 1) = \prod_{k=1}^n k = n!$$

Thus, the product is:

$$\prod_{(i,j) \in \lambda} h(i, j) = (n + 1)!n!$$

#### Step 3: Apply Hook-Length Formula

The number of SYT is:

$$f^{(n,n)} = \frac{(2n)!}{(n+1)!n!}$$

which conforms with the expression of Catalan number  $C_n$

Thus, the number of SYT of shape  $(n, n)$  is  $C_n$ .

### 6.3. Summarization

A pivotal case arises from the rectangular Young diagram of shape  $\lambda = (n, n)$ . This diagram consists of two rows of equal length  $n$ , giving a total of  $2n$  cells. The structure and properties of Standard Young Tableaux of this specific shape are intimately related to the Catalan numbers, as its enumeration yields the familiar formula for  $C_n$ [11].

### 7. Conclusion

In this work, it could be found that the application of Catalan numbers is exactly a process of applying the recursive formula, which is also a way of mapping complex problem into a known formula of Catalan numbers. **Moreover**, with the generalization, the Fuss Catalan numbers can extend the applications in wider cases. For instance, the  $j$ -ary trees and even problems in group theory. Under such condition, Catalan numbers literally provide a great convenience in solving problems in enumerative combinatorics, with an inspiration of mapping being shown well.

### References

- [1] S. Roman, Introduction to catalan numbers. Cham: Birkhauser, 2015.
- [2] J. Auger, Orbits of the Dissected Polygons of the Generalized Catalan Numbers. LAP Lambert Academic Publishing, 2012.
- [3] J.-C. Aval, "Multivariate Fuss–Catalan numbers," Discrete Mathematics, vol. 308, no. 20, pp. 4660–4669, Oct. 2008, doi: <https://doi.org/10.1016/j.disc.2007.08.100>.
- [4] I. M. Gessel, "Lagrange inversion," Journal of Combinatorial Theory, Series A, vol. 144, pp. 212–249, Nov. 2016, doi: <https://doi.org/10.1016/j.jcta.2016.06.018>.
- [5] D. Merlini, R. Sprugnoli, and M. C. Verri, "Lagrange Inversion: When and How," Acta applicandae mathematicae, vol. 94, no. 3, pp. 233–249, Dec. 2006, doi: <https://doi.org/10.1007/s10440-006-9077-7>.
- [6] I. Pak, "Reduced decompositions of permutations in terms of star transpositions, generalized Catalan numbers and  $k$ -ARY trees," Discrete Mathematics, vol. 204, no. 1–3, pp. 329–335, Jun. 1999, doi: [https://doi.org/10.1016/s0012-365x\(98\)00377-x](https://doi.org/10.1016/s0012-365x(98)00377-x).
- [7] T. Mansour and Alek Vainshtein, "Restricted 132-Avoiding Permutations," Advances in Applied Mathematics, vol. 26, no. 3, pp. 258–269, Apr. 2001, doi: <https://doi.org/10.1006/aama.2000.0719>.
- [8] T. Mansour and M. Shattuck, "Pattern-Avoiding Set Partitions and Catalan Numbers," Electronic Journal of Combinatorics, vol. 18, no. 2, Feb. 2012, doi: <https://doi.org/10.37236/2048>.
- [9] P. Alexandersson, S. A. Fufa, F. Getachew, and D. Qiu, "Pattern-avoidance and Fuss-Catalan numbers," arXiv (Cornell University), vol. 26, no. 23.4.2, Jan. 2022, doi: <https://doi.org/10.48550/arxiv.2201.08168>.
- [10] P. Drube, "Set-Valued Tableaux & Generalized Catalan Numbers," Australas. J Comb., vol. 72, pp. 55–69, Jan. 2018.
- [11] J. Buontempo and B. Hopkins, "Tableau cycling and Catalan numbers.," Integers, vol. 7, no. 1, Jan. 2007.