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Chain-k Mappings: A Combinatorial and Spectral Approach to Analyzing Complete Mappings

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Abstract. This paper investigates bijections on a group G that arise from products of the form $f(x)g(x)$, a problem that is centrally connected to the concept of a **complete mapping**. We introduce the notion of a **"chain-k mapping"** to analyze the structure of certain full-cycle permutations and explore the relationship between complete mappings and the spectral properties of their associated permutation matrices. Key results include a proof that the cyclic group $\mathbb{Z}/n\mathbb{Z}$ admits a complete mapping if and only if n is odd, and a characterization of the cycle structures of permutations σ for which the maps $id+\sigma$ and $id-\sigma$ are automorphisms of a specific vector subspace. This work establishes a connection between combinatorial group theory and the eigenvalue theory of permutation matrices, viewed through the novel lens of **"(-1)-elliptic elements".** We will begin with the motivation that led to this study, and look at problems including *elliptic elements* and *complete mappings over finite fields*.

Keywords: Complete Mappings, Cyclic Groups, Roots of Permutations, Elliptic Elements

1. Introduction

We begin with a seemingly irrelevant equation:

$$\sin x + \arcsin x = \sin 1 + \frac{\pi}{2}$$

which has a unique solution $x = 1$. The uniqueness of this root is a key motivation: in the context of continuous functions, this is a result of both $\sin$ and $\arcsin$ being increasing functions hence having a bijective sum over their domain, but when one moves the perspective to *all functions*, which include discontinuous functions and those over *finite sets*, the complexity of the problem increases exponentially.

In a generalized background with some modifications, we are concerned about the self bijections f, g of a set S (which could be a vector space, a group, a field, etc.) such that for each element x, h defined as $h(x) = f(x) * g(x)$ is still a bijection from S to $h(S)$. An interesting question (also an important motivation) derived from this is:

> *(1) Let A be a set of positive integers from 1 to n. Let $\sigma \in S_n$ act on A such that $a \mapsto \sigma(a)$ for $a \in A$. Which σ are such that the function $\tau : A \to \tau(A)$ defined as $\tau(a) = \sigma(a) + \sigma^{-1}(a)$ is a bijection?*

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(1) is a direct result of the modification for the beginning motivation. It provides some insights that help us link to concepts that are more well-studied.

First, notice that the image of f is not A itself: that is, addition is not closed in A. Hence a possible way to extend or reconstruct this problem is to let A be a (here, abelian or cyclic) group.

The second implication of this problem leads directly to the main theme of the paper. The key is to intuitively notice that, it is possible to remodify τ to $\tau_1(x) = \sigma(x) + x$, such that the σ that induces bijective τ_1 have similar properties to those that induce bijective τ. This leads to the core idea of *complete mappings.*

In reality, the actual journey from the trigonometric equation to complete mappings took much longer and had many detours. The Introduction only provides a very quick overview of how this was done. We therefore formally begin the paper by defining *complete mappings* and some of its general properties, and then look at specific problems.

2. Complete Mappings

Definition 1.1 Let G be a group. A function $\phi : G \to G$ is called a *complete mapping of* G if both ϕ and $\phi * id : g \mapsto g\phi(g)$ are bijections of G.

Definition 1.2 A group G is said to be *admissible* if there exists a complete mapping for G; otherwise it is said to be *non-admissible*. [1]

Definition 1.3 For a group G, denote the set of all complete mappings of G as C, and bijections of G as B.

Theorem 1.1 *If G is an admissible group, there exist an ordering of the elements of G such that* $g_1 g_2 \dots g_n = 1$. [1]

Due to the nature of the operation $+$ in the motivation, the study concerns such groups G that are *abelian*, for example, $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{F}_p$, $\mathbb{F}^n$. Hence the following theorem is sufficient:

Proposition 1.1 *If G is an admissible abelian group,*

$$\prod_{g \in G} g = 1.$$

Proof. Let $\varphi = \phi * id$. Since $\varphi \in B$ by definition, $\varphi(G) = G$. Hence

$$\prod_{\varphi(g) \in G} \varphi(g) = \prod_{g \in G} g$$
$$\prod_{g \in G} g\phi(g) = \prod_{g \in G} g \quad \text{and since } \phi \in B,$$
$$\prod_{g \in G} g\phi(g) = (\prod_{g \in G} g)^2 = \prod_{g \in G} g \quad \text{giving}$$
$$\prod_{g \in G} g = 1.$$

An extension of Proposition 1.1 involves defining a generalized complete mapping. We first define this.

Definition 1.4 Let G be a group and $\psi : G \to G$ a bijection of G. Then a bijection $\phi : G \to G$ is called a ψ*-complete mapping* of G if $\tau : G \to G$ defined by $\tau(g) = \psi(g)\phi(g)$ is also a bijection of G. [2]

By a similar proof to Proposition 1.1, we can show that there exists some ψ-complete mapping for G only if $\prod_{g \in G} g = 1$. This shows that this is an important necessary condition for all types of "binary complete mappings", including the case $f * f^{-1}$, which implies how this is *almost* essentially the same as $f * id$.

The reason why the mappings $f * f^{-1}$ and $f * id$ are not *completely* the same in essence is due to the concept of square roots in the symmetric group. Note that $h(x) = f^{-1}(x)f(x)$ is a bijection if and only if $h(f^{-1}(x)) = xf^2(x)$ is a bijection if and only if f^2 is a complete mapping. If we want to establish a bijective correspondence between the set of f such that $f * f^{-1}$ are bijections and complete mappings, we have to prove that every complete mapping has a unique square root. This is generally not true, and only holds specific cases, for example when we assume that these mappings correspond to full cycles in S_{Z_n} (Corollary 1.1 will help prove this). We will now restrict our focus on G as a *cyclic group*.

In fact, cycle types are significant for complete mappings of finite groups. Full cycles and cycles of order n will be special cases and we will look at them later. Before that, more results will be presented:

Corollary 1.1 *$\mathbb{Z}/n\mathbb{Z}$ is an admissible group if and only if n is odd.*

Proof. For the group $\mathbb{Z}/n\mathbb{Z}$, since the operation is addition, Proposition 1.1 implies that if it is admissible, then the sum of all elements must be congruent to 0 modulo n:

$$\sum_{x=0}^{n-1} x = \frac{n(n-1)}{2} \equiv 0 \pmod{n}$$

This congruence holds if and only if $\frac{n(n-1)}{2}$ is an integer multiple of n, which implies that $\frac{n-1}{2}$ must be an integer. This is only possible if $n-1$ is an even number. Therefore, a necessary condition for the group to be admissible is that n must be odd.

For the opposite direction, assume n is odd. Let $\phi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ be defined such that $x \mapsto x+1 \pmod{n}$. This mapping is a full cycle and thus a bijection. We then examine the map $\eta(x) = x+\phi(x)$:

$$\begin{aligned} x + \phi(x) &\equiv y + \phi(y) \pmod{n} \\ \iff 2x + 1 &\equiv 2y + 1 \pmod{n} \\ \iff 2x &\equiv 2y \pmod{n} \end{aligned}$$

Since n is odd, $\gcd(2, n) = 1$, which allows for cancellation of the factor 2. This gives $x \equiv y \pmod{n}$, proving that η is a bijection. As both ϕ and η are bijections, ϕ is a complete mapping, and $\mathbb{Z}/n\mathbb{Z}$ is admissible.

There is a general result, though not iff, for finite groups.

Theorem 1.2 (Hall-Paige Theorem, once a conjecture) *A finite group G admits a complete mapping if and only if its Sylow-2 subgroup is either trivial or noncyclic. (Hence if it has odd order, it admits complete mappings)* [3]

If we consider only full cycle permutations of the cyclic group, these have square roots if and only if the group has odd order, so all complete mappings corresponding to full cycle permutations in S_{Z_n} have unique square roots, thus only in this case, complete mappings are bijective images of bijections $f * f^{-1}$.

A sufficient condition for existence of unique kth roots for full cycles n is that k and n are coprime. The proof is given below:

Proposition 1.2 *For a symmetric group S_n, if $(n, k) = 1$, a full cycle permutation has a unique kth root.*

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Proof. If $(n, k) = 1$, by the Euclidean Algorithm [4], there exist integers x, y such that

$$ak + bn = 1, \quad \text{which means that}$$
$$ak \equiv 1 \mod n.$$

Let σ be an arbitrary n-cycle in S_n, and suppose there exists $\tau \in S_n$ such that $\tau^k = \sigma$. Now consider the permutation $\tau = \sigma^a$. Then $\tau^k = \sigma^{ka} = \sigma^1 = \sigma$, so τ is indeed a kth root of σ. Suppose we have τ' as another kth root of σ. Then τ' must also be a n-cycle and $\sigma \in \langle\tau'\rangle$ so $\langle\sigma\rangle \leq \langle\tau'\rangle$. Since both subgroups have the same order, they are exactly the same. As a result, $\tau' = \sigma^j$, so $(\sigma^j)^k = \sigma^{jk} = \sigma$, meaning that $jk \equiv 1 \pmod n$. Given that $(k, n) = 1$, k has an inverse in $\mathbb{Z}/n\mathbb{Z}$, so there is only one unique root to the equation $xk \equiv 1 \pmod n$, which forces $j \equiv a$ hence $\tau = \tau'$. Therefore, the full cycle σ has a unique kth root τ.

Now if $|G|$ is odd and $\sigma \in S_G$ is a full cycle complete mapping of G, σ has a unique square root since 2 is coprime to all odd numbers. Moreover, if $|G|$ is prime, σ has unique kth roots for $k \not\equiv 0 \pmod p$. Notice that if we link this to complete mappings, it implies that there are bijective correspondence between every mapping such that their kth power are complete mappings. We give a definition for this type of mapping for clarity:

Definition 1.5 A permutation ϕ of a finite group G is called a *kth-complete mapping* if ϕ^k is a complete mapping of G. For ϕ being a m-cycle, we call ϕ a *m-chain-k mapping* of G. If $m = |G|$ we call it a *chain-k mapping.*

Example 1.1. Let us provide an example of a chain-k mapping. Consider the additive group $G = \mathbb{Z}/5\mathbb{Z}$. Let σ be the full-cycle permutation $\sigma = (0\ 1\ 2\ 3\ 4)$. We can show that σ is a *chain-2 mapping*.

According to the definition, we must verify that the map $\eta : x \mapsto x + \sigma^k(x)$ for $k = 2$ is a bijection on G. First, we compute the permutation σ^2, which maps each element to the element two steps forward in the cycle: $\sigma^2 = (0\ 2\ 4\ 1\ 3)$.

Now, we compute the image of each element under $\eta(x) = x + \sigma^2(x)$:

$$\begin{aligned}
\eta(0) &= 0 + \sigma^2(0) = 0 + 2 = 2 \\
\eta(1) &= 1 + \sigma^2(1) = 1 + 3 = 4 \\
\eta(2) &= 2 + \sigma^2(2) = 2 + 4 = 1 \pmod 5 \\
\eta(3) &= 3 + \sigma^2(3) = 3 + 0 = 3 \\
\eta(4) &= 4 + \sigma^2(4) = 4 + 1 = 0 \pmod 5
\end{aligned}$$

The image set under η is $\{2, 4, 1, 3, 0\}$, which is a permutation of the elements of G. Since the map η is a bijection, we conclude that $\sigma = (0\ 1\ 2\ 3\ 4)$ is indeed a chain-2 mapping of $\mathbb{Z}/5\mathbb{Z}$.

It is then clear that for k coprime to n, there is a bijective correspondence between all full cycle kth-complete mappings and all full cycle complete mappings due to the existence of unique kth roots, hence bijective correspondence between every kth-complete mappings ($k = 1$, which are complete mappings, included). In a setting where G has prime order, we do not miss any non-zero k (modulo p). For G being abelian groups, if they have p order, they are also *simple groups* such that they are the buiding blocks of all finite abelian groups, and are isomorphic to cyclic groups of prime order. The harmony achieved with this assumption drives further investigation into specific cases for *(1) full cycle complete mappings* and *(2) abelian G of prime order (Z_p),and finite fields of prime order,* $\mathbb{F}_p$.

There is a very intuitive motivation for naming *chain-k mappings*. Consider arranging all the elements g_i of a group in the following way (see Figure 1):

Then every such cycle corresponds to a full cycle $\sigma = (g_1\, g_2\, g_3\, \ldots\, g_n) \in S_G$. We can see that σ is a complete mapping if and only if $g_i g_{i+1}$ gives distinct results for different i. If σ is a chain-k

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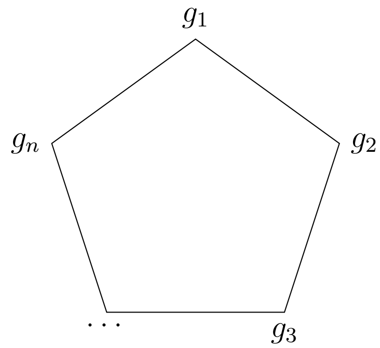

Figure 1: A visualization of a full-cycle permutation $\sigma = (g_1\, g_2\, g_3\, \ldots\, g_n)$ on a group G. The permutation σ is a *chain-1 mapping* (a standard complete mapping) if the products of adjacent elements $(g_i g_{i+1})$ are all distinct. More generally, it is a *chain-k mapping* if the products of elements k steps apart along the cycle are distinct.

mapping if and only if, in the clockwise direction, multiplying two elements with k edges between them gives distinct results for each pair of elements. k indicates the number of edges between each pair of elements and *chain* indicates that we are ordering the elements of G such that they form a chain like structure (cycle). Also note that for an abelian group, since elements commute, multiplying assigned pairs in the anticlockwise direction also give distinct products for each pair, so for *abelian* G, if ϕ is a chain-k mapping, so is its inverse.

We can turn each chain-k mapping into a chain-1 mapping (a full cycle complete mapping), by rearranging the graph and let every g_i, g_{i+k} be consecutive terms, *provided* $(k, n) = 1$ (this should look quite straightforward from the geometric image), else we would get multiple m-chain graphs (decomposing the original graph). The graph does not decompose under this transformation for all *non-trivial* chain-k mappings, that is, $k \not\equiv 0 \pmod{n}$, when $n = p$, so prime orders is a desired property that keeps things clean and closed. This is yet another amazing case of "irreducibility" and constancy for prime items.

Question (2), still unsolved, results from this exploration:

> *For a finite field of odd order p, how many chain-1 mappings on its additive group are there?*

The available numbers are: 2, 4, 48, and 5400 for primes 3, 5, 7, 11 respectively. If we only look at the complete mappings of $\mathbb{Z}/n\mathbb{Z}$, this is 360 for the odd number 9. Since every number is divisible by $p - 1$ it is almost confirmed that there is a pattern. This question is not yet pursued due to time constraints, but some simple results are available. First, recall the fact that every permutation of a finite field can be expressed using a polynomial in $\mathbb{F}_q[x]$. Theorem 1.3 is built upon this.

Theorem 1.3 [1] *If $f(x)$ is a complete mapping polynomial of $\mathbb{F}_q$, then so are the following polynomials:*

(i) $f(x + a) + b$ *for all* $a, b \in \mathbb{F}_q$
(ii) $af(a^{-1}x)$ *for all* $a \in \mathbb{F}_q$
(iii) *any polynomial representing the inverse mapping of* $c \in \mathbb{F}_q \to f(c)$

Moreover, some small but possibly inspiring structures have been found. For a complete mapping ϕ let $\eta = \phi + id$. η is naturally another permutation of the field. An interesting experiment to do is

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to see what happens when we conjugate ϕ by η_0. Note that this gives a new p-cycle $\psi_1 = \eta_0\phi\eta_0^{-1}$. For some ϕ, this is another complete mapping, and if we repeat the conjugation process by letting $\psi_{i+1} = \eta_i\psi_i\eta_i^{-1}$, where $\eta_i = \psi_i + id$, we get more complete mappings. The process eventually returns to ϕ to create a collection of cycles that are all complete mappings. This works for polynomials $x + d$. Most of the time, the process stops after two iterations. ϕ are usually not complete mappings. However, if we ignore the permutation structure of the cycle and continue to apply the algorithm on its entries, so that for $(a_1\, a_2 ... a_p)$, $a_i \mapsto a_i + a_{i+1}$, within an entire repetition before we get back to ϕ, there are some disjoint pieces of permutations that are generated on the way, in some cases complete mappings.

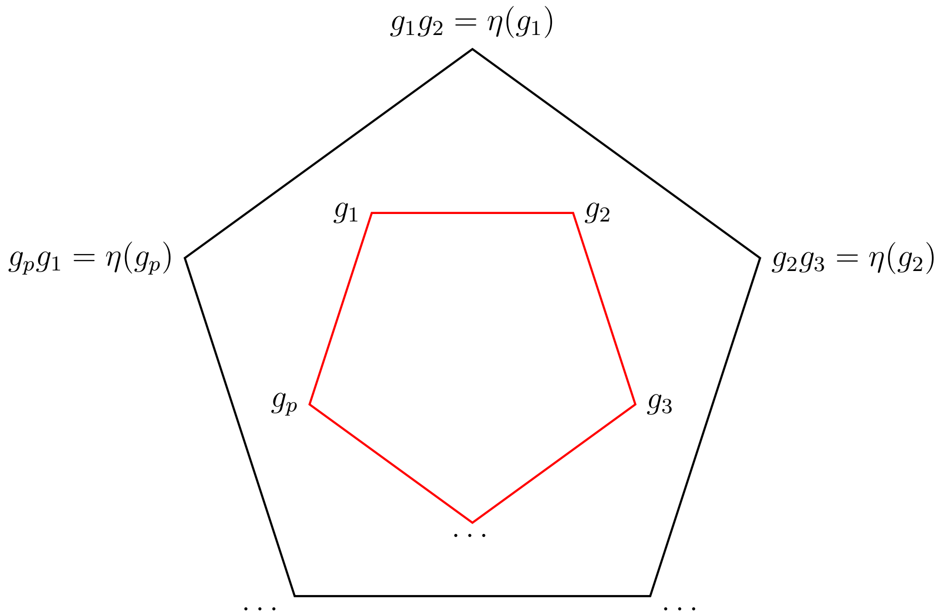

Figure 2: Visual representation of the mapping transformation

3. Elliptic Elements

This is a section where we examine complete mappings that are *homomorphisms*. There are some concrete results for some complete mappings on vector spaces $\mathbb{C}^n$. It is almost only reasonable to consider such complete mappings that are *isomorphisms* of a vector space, so that a sensible structure is preserved. This gives rise the following question:

> *(3) Let H be a hyperplane in $\mathbb{C}^n$, such that $H = \{(x_1, x_2, \ldots, x_n) \mid \sum x_i = 0\}$. Let the symmetric group S_n act on H by permuting the coordinates of the vector $\mathbf{a} \in A$. Which $\sigma \in S_n$ are such that $\tau = id + \sigma$ defined by the map $\mathbf{v} \mapsto \mathbf{v} + \sigma \cdot \mathbf{v}$ is an isomorphism of H?*

Before looking at the case of $id + \sigma$, it is helpful to see what happens when $\tau = id - \sigma$. We reasonably construct a linear representation (which by definition is a homomorphism)

$$\rho : S_n \to GL_n(\mathbb{C}) \quad s.t. \quad \sigma \mapsto P_\sigma$$

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where P_σ is the permutation matrix for σ, and

$$\sigma \cdot \mathbf{a} = P_\sigma \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} a_{\sigma^{-1}(1)} \\ \vdots \\ a_{\sigma^{-1}(n)} \end{pmatrix}.$$

From this setting it is then apparent that τ is necessarily a *linear map* (hence a homomorphism), so saying that τ is an isomorphism is equivalent to saying that $null\,\tau = 0$, hence for $\mathbf{v} \neq \mathbf{0}$, $\mathbf{v} - \sigma \cdot \mathbf{v} \neq \mathbf{0}$ so $\sigma \cdot \mathbf{v} \neq \mathbf{v}$.

Note that permuting the entries (essentially basis vectors) leaves H as an invariant subspace of $\mathbb{C}^n$, since the sum of these entries is always 0 no matter how they are arranged. $\sigma \cdot \mathbf{v} \neq \mathbf{v}$ is equivalent to $P_\sigma \mathbf{v} \neq \mathbf{v}$ for $\mathbf{v} \neq \mathbf{0}$. This implies that σ is in fact a *elliptic element*: which is to say, that its permutation matrix P_σ contains at most *one* eigenvalue of 1.

This equivalence is quite straightforward. Since by definition all P_σ are diagonalizable, it must have n linearly independent eigenvectors $\{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}\}$, with corresponding eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n \mid \lambda_i \in \mathbb{C}\}$. The eigenvectors span V, so we can write any vector as a linear combination of the eigenvectors, which means that for $\mathbf{0} \neq \mathbf{v} \in H$, and $c_i \in \mathbb{C}$,

$$\begin{aligned} \mathbf{v} &= c_1\mathbf{e_1} + c_2\mathbf{e_2} + \cdots + c_n\mathbf{e_n} \quad \not\forall c_n = 0 \\ P_\sigma \mathbf{v} &= c_1\lambda_1\mathbf{e_1} + c_2\lambda_2\mathbf{e_2} + \cdots + c_n\lambda_n\mathbf{e_n} \quad so \\ \mathbf{v} - P_\sigma\mathbf{v} &= c_1(1-\lambda_1)\mathbf{e_1} + \cdots + c_n(1-\lambda_n)\mathbf{e_n}. \end{aligned}$$

Since the orthogonal complement of H, $H^\perp = span\,(\mathbf{a}^\perp) = span\,((1,1,1,\ldots,1))$ has eigenvalues of 1 for each vector, if σ is *elliptic* this would mean that $\forall \lambda_i \neq 1$, meaning that at least for one i, $c_i(1-\lambda_i)\mathbf{e_i} \neq \mathbf{0}$, hence $\mathbf{v} - P_\sigma\mathbf{v} \neq \mathbf{0}$ giving $(I - P_\sigma)\mathbf{v} \neq \mathbf{0}$ (so that $(id - \sigma)\cdot\mathbf{v} \neq \mathbf{0}$ for $\mathbf{v} \neq \mathbf{0}$), which is also to say that $(id - \sigma)$ is a bijection or an *automorphism* of the subspace H. The implication goes the opposite way around (which is also a similar simple proof), so we arrive at a useful equivalence: [5]

$$\begin{aligned} (1-\sigma)\ \textit{is an automorphism} &\iff \sigma\ \textit{does not fix any non-zero vector in } H \\ &\iff P_\sigma\ \textit{has at most 1 eigenvalue of 1} \end{aligned}$$

Now by exploiting the properties of the homomorphism ρ, we can quickly deduce a result for the case $(id - \sigma)$. Since all permutation can be written as a product of disjoint cycles, let $\sigma = c_1c_2c_3\ldots c_n$, where c_i are disjoint cycles. Then as a result of homomorphism,

$$\rho(\sigma) = \rho(c_1 \ldots c_n) = \rho(c_1)\rho(c_2)\ldots\rho(c_n).$$

Also note that the fibers above the image of S_n under ρ are all single elements in S_n. As a result, ρ is essentially an *isomorphism* from S_n to its image in $GL_n(\mathbb{C})$, meaning that for each $\tau \in S_n$, $|\tau| = |P_\tau|$. If $|\tau| = k$, then $|P_\tau|^k = I$ so if μ is an eigenvalue of P_τ, and $\mathbf{v}$ is an eigenvector under the action of τ,

$$\begin{aligned} P_\tau^k\mathbf{v} &= I\mathbf{v} \\ \mu^k\mathbf{v} &= \mathbf{v} \\ \mu^k &= 1 \end{aligned}$$

so μ is a kth root of unity. As a result, each c_i with a length of l_i has l_i distinct eigenvalues in its corresponding permutation matrix $\rho(c_i)$. If we fix a basis of spanning eigenvectors of $\rho(\sigma)$, performing

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each $\rho(c_i)$ (in any order due to commutativity) is just assigning eigenvalues to the set of l_i eigenvectors unique to c_i. As a result, by applying every $\rho(c_i)$ one obtains the eigenvalues for $\rho(\sigma)$ as the collection of all eigenvalues for each $\rho(c_i)$. Immediately, we see that if there is more than one c_i, there is more than *one* eigenvalue of 1 since every permutation matrix must have 1 as an eigenvalue as it is always a root of unity. This means that there can only be one cycle in the cycle decomposition of σ, implying that it is a $n - cycle$. So at the first stage we have the result:

$$\begin{aligned} & (id - \sigma) \textit{ is an automorphism of } H \\ \iff & \sigma \textit{ is elliptic in H} \\ \iff & \rho(\sigma) \textit{ has at most one eigenvalue of 1} \\ \implies & \sigma \textit{ is a n-cycle.} \end{aligned}$$

3.1. *(-1)-Elliptic* Elements

The motivation for defining *(-1)-elliptic* elements in S_n comes from the observation that a similar method to the previous section can be applied to find σ such that $(id + \sigma)$ is an automorphism of H. We rephrase this into $(I + P_\sigma)\mathbf{v} \neq \mathbf{0}$, for $\mathbf{v} \neq \mathbf{0}$. Then it follows, almost identical to the case for elliptic elements, that this happens *if and only if* there are **no** eigenvalues of P_σ equal to -1. We disregard the phrasing of *"no more than one eigenvalue of..."* used in the last section, since the orthogonal vector to H, $(1, 1, \ldots, 1)$, does not have an eigenvalue of -1 under any permutation, since its eigenvalue is 1 under every permutation. Hence the question becomes: for which σ does P_σ have *no* eigenvalue of -1?

We only have to notice that, if a cycle has an even number of terms, it must contain -1 as an eigenvalue, as it is an even root of unity. As a result, saying that P_σ has no eigenvalue of -1 is equivalent to saying that no matrix in the corresponding "matrix decomposition" of $\rho(\sigma)$ to the cycle decomposition of σ should have an eigenvalue of -1, which is the same as saying that no cycle c_i should have an even number of terms. The answer is therefore:

$$\begin{aligned} & (id + \sigma) \textit{ is an automorphism of H} \\ \iff & \textit{Every cycle in the cycle decomposition of } \sigma \textit{ is even.} \end{aligned}$$

4. Conclusion

The core contribution of this paper is the revelation of the rich combinatorial and geometric structures underlying the algebraic concept of complete mappings. By introducing "chain-k mappings," we translate an abstract algebraic property into a more intuitive 'step-wise' relationship on a cyclic graph. This geometric intuition, in turn, guides our extension of the problem into the realm of linear algebra and the spectral properties of permutation matrices. Ultimately, this work demonstrates that a single problem concerning bijections on a group can be understood from three distinct yet unified perspectives: the algebraic (group theory), the combinatorial (cycle structures), and the linear (eigenvalues).

The framework of "chain-k mappings" introduced herein serves as an effective new tool for investigating full-cycle complete mappings. It successfully transforms a complex algebraic condition—that the map $x \mapsto x\phi(x)$ is a bijection—into a more tangible and verifiable combinatorial criterion. This model is particularly powerful for groups of prime order, where its "irreducibility" simplifies the problem and reveals a potential bijective correspondence between chain-mappings of different orders. This not only facilitates theoretical analysis but also provides a clearer path for the computational search and enumeration of such maps.

Perhaps the most significant outcome of this investigation is the identification of a challenging open problem: the exact enumeration of chain-1 mappings on a finite field of prime order p. Our initial computational data reveals an intriguing pattern—the count appears to be divisible by $p-1$—strongly suggesting a deep connection to the structure of the multiplicative group $\mathbb{F}_p^*$. Key future research tasks will involve two main directions: first, developing efficient computational algorithms to obtain data for larger primes, which will be crucial for pattern recognition and conjecture formulation; and second, seeking a theoretical framework from algebraic number theory or combinatorial design to explain this phenomenon. A solution to this problem would likely deepen our understanding of the structure of permutation polynomials over finite fields.

In conclusion, this research not only delineates key properties of complete mappings but, more importantly, opens new avenues for exploring bijective structures on finite groups by building a bridge between their algebraic, combinatorial, and linear-algebraic aspects.

References

[1] Paige, L. J. (1951) ‘Complete Mappings of Finite Groups’, Pacific Journal of Mathematics, vol. 1, no. 1, November 1951, pp. 111–116. Available at: Pacific Journal of Mathematics website

[2] Niederreiter, H. & Robinson, K. H. (1982) ‘Complete mappings of finite fields’, Journal of the Australian Mathematical Society, Series A, 33(2), pp. 197–212.

[3] Hall, M. and Paige, L. J. (1955) ’Complete mappings of finite groups’, Pacific Journal of Mathematics, 5(4), pp. 541–549.

[4] Dummit, D. S. and Foote, R. M. (2004) Abstract Algebra, 3rd edn, John Wiley & Sons, Hoboken, NJ. ISBN 978-0471433347.

[5] Axler, S. J. (2023) Linear Algebra Done Right, 4th edn, Undergraduate Texts in Mathematics, Springer, New York. ISBN 978-3031410253.