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Research on the Proof and Application of the Orbit-Stabilizer Theorem

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Abstract. Group theory is a very important concept in mathematics with many interesting theories that have been widely applied in other areas of mathematics. As one of the fundamental tools in abstract algebra, it provides a unifying language for studying symmetries, structures, and transformations, making it central to both theoretical and applied mathematics. This paper proves the orbit stability theorem based on the theory of group actions. Then, this paper introduces the application of the orbit stabilizer in other parts of mathematics and its full proof. Among these theorems, compared with other proof methods, the orbit stabilizer theorem is more concise and easier to understand. These examples show the wide application of the orbit stability theorem in mathematics, proving its practicality. Furthermore, the theorem serves as a foundation for exploring topics such as combinatorics, number theory, and geometry, where orbit-stabilizer arguments simplify otherwise complex counting and classification problems. In this way, the study highlights how group theory not only develops its own framework but also contributes essential insights to broader mathematical investigations.

Keywords: Group theory, group action, orbital stabilizer, cauchy theorem, burnside counting theorem.

1. Introduction

In the abstract algebra, group theory is very important. The early 19th century, French mathematician Évariste Galois in post-duel manuscript, he first introduced the concept of a group. He perceived the fundamental nature of the impossibility of solving the fifth-degree equation, is independent with coefficient, but the symmetrical structure of the roots. Later, based on the group, concepts such as ring, field, ideal, trace and horm were also derived. These concepts have occupied a very important position in abstract algebra and applied mathematics.

Joseph Lagrange in his paper first proposed the Lagrange theorem in his paper [1], this theorem implies some of the thought in Orbit-Stabilizer. Subsequently, in Augustin-Chevalier's research on permutation groups, the concept Stabilizer was clearly defined for the first time [2]. Finally, in the book published by William Burnside in 1911, the theorem was clearly stated, and the final proof was provided [3]. Apart from within the field of mathematics itself, group theory has played a significant role in many other areas as well, such as in physics and chemistry in the natural sciences. Take physics as an example, in nonrelative quantum mechanics, the interior of non-relativistic hydrogen

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atoms has the $O(4)$ symmetry group, and this group and used to relating different bound states [4]. This author also introduces into a bigger group $O(1,4)$, the approach used here can be extended to obtain the special unitary representations of non-compact groups [4].

This paper is written in 3 sections. The first section is about the proof of Orbit-Stabilizer, this paper will proof some of the basic theorem of group theory and use them to proof and define the Orbit-Stabilizer. The next section is the introduction about Orbit-Stabilizer, and how important abstract algebra and applied mathematics. The last section is about some famous applications of Orbit-Stabilizer, and some examples of this theorem.

2. Background knowledge and theorem proof

2.1. Group action

Group action is one of the very important concepts in mathematics [5, 6]. It is a fundamental basis for many renowned theories and other concepts in group theory. This includes the orbital stabilizer that this paper will discuss later.

Definition:

Assuming G is a group, and X is a finite set, a group action is a mapping

$$G \times X \to X,\ (g, x) \mapsto g^{*}x \tag{1}$$

The ordered pair (g, x) mapped into $g^{*}x$, from $G \times X$ to X.

And this should meet the following conditions

Identity:

$$\forall x \in X, e^{*}x = x\ (e\ is\ the\ identity\ of\ G) \tag{2}$$

Compatibility:

$$\forall g, h \in G,\ x \in X,\ g^{*}(h^{*}x) = (gh)^{*}x \tag{3}$$

2.2. Orbital stabilizer

Orbital stabilizer theorem [7, 8]:

Assuming G is a group acting on the finite set X.

$$\forall x \in X,\ Orb(x) = \{\tau^{*}x\tau \in G\} \tag{4}$$

$$\forall x \in X,\ Stab(x) = \{\tau \in G\tau^{*}x = x\} \tag{5}$$

$Orb(x)$ denotes the orbit of x, $Stab(x)$ denote the stabilizer of x by G. Then

$$|G| = |Orb(x)| \bullet |Stab(x)| \tag{6}$$

Process of proof:

Identity

$$\forall x \in X,\ ex = x \tag{7}$$

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$$\forall\tau \in G,\ \tau\left(ex\right) = \tau x \tag{8}$$

Thus $e \in Stab(x)$.

Inverse

And because $\forall x \in X,\ Stab\left(x\right) = \{\tau \in G\tau^{*}x = x\}$, if $\tau x = x$, $\tau^{-1}\tau x = ex = x$, so $\tau^{-1} \in Stab\left(x\right)$.

Closure

$\forall\tau, h \in Stab\left(x\right)$, $\tau hx = \tau\left(hx\right) = x$, so $\tau h \in Stab(x)$.

Finally, by the definition of group action, it must satisfy the associative law.

So $Stab(x)$ is a group.

And because $Stab(x) \leq G$, so $Stab(x)$ is a subgroup of G .

Defining a mapping ϕ

$$\phi : G/Stab(x) \to Orb(x) \tag{9}$$

$$\phi\left(\tau\ Stab\left(x\right)\right) = \tau \bullet x \tag{10}$$

Assuming $\tau\ Stab\left(x\right) = \tau^{'}Stab\left(x\right)$, based on the properties of the cosets, it same as $\tau^{-1}\tau^{'} \in Stab\left(x\right)$. Because of this, $\tau^{-1}\tau^{'}x = x$.

$$\tau\left(\tau^{-1}\tau^{'}x\right) = \tau x \tag{11}$$

$$\left(\tau\tau^{-1}\tau^{'}\right)x = \tau x \tag{12}$$

$$\tau^{'}x = gx \tag{13}$$

$$\phi\left(\tau\ Stab\left(x\right)\right) = \tau x = \tau^{'}x = \phi\left(\tau^{'}\ Stab\left(x\right)\right) \tag{14}$$

So, the mapping ϕ is well-defined.

$$\forall y \in Orb\left(x\right),\ \exists\tau \in Stab\left(x\right),\ y = \tau x \tag{15}$$

So, $\phi\left(\tau\ Stab\left(x\right)\right) = \tau \bullet x = y$, ϕ is surjective.

Assuming $\phi\left(\tau\ Stab\left(x\right)\right) = \phi\left(\tau^{'}\ Stab\left(x\right)\right)$, it means $\tau x = \tau^{'}x$.

$$\tau^{-1}\left(\tau x\right) = \tau^{-1}\left(\tau^{'}x\right) \tag{16}$$

$$\left(\tau^{-1}\tau\right)x = \left(\tau^{-1}\tau^{'}\right)x \tag{17}$$

$$ex = \left(\tau^{-1}\tau^{'}\right)x \tag{18}$$

$$x = \left(\tau^{-1}\tau^{'}\right)x \tag{19}$$

So, $\tau^{-1}\tau^{'} \in Stab\left(x\right)$, $\tau\ Stab\left(x\right) = \tau^{'}Stab\left(x\right)$.

ϕ is injective. And because ϕ is both surjective and injective, it is a bijection.

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Because the mapping $\phi : G/Stab(x) \to Orb(x)$ is a bijection, so $|G/Stab(x)| = |Orb(x)|$. Based on the Lagrange 's Theorem,

$$|G/Stab(x)| = \frac{|G|}{|Stab(x)|} = |Orb(x)| \tag{20}$$

$$|G| = |Orb\,(x)| \bullet |Stab\,(x)| \tag{21}$$

3. Applications

3.1. Cauchy theorem

Theorem description:

Assuming G is a finite group and p is a prime number. If $p|\,|G|$, then there must be an element $\tau \in G$ that satisfy the following conditions:

$$\tau \neq e \tag{22}$$

$$\tau^p = e \tag{23}$$

The order of τ is p

Process of proof:

Defining a set $X = \{(\tau_1,\ \tau_2, \ldots, \tau_p) \in G^p \tau_1\tau_2 \ldots \tau_p = e\}$, then $\tau_p = (\tau_1\tau_2 \ldots \tau_{p-1})^{-1}$, so, $|x| = |G|^{p-1}$, and because $p|\,|G|$, thus $p|\,|X|$.

Next defined the group action, let cyclic group $\mathbb{Z}/p\mathbb{Z} = \{0,1,2,\ldots,p-1\}$ act on the set X .

$\forall n \in \mathbb{Z}/p\mathbb{Z}$, defined that $n \bullet (\tau_1,\ \tau_2, \ldots, \tau_p) = (\tau_{n+1},\ \tau_{n+2}, \ldots, \tau_p, \tau_1 \ldots, \tau_n)$

Identity: when $n = 0$, $n \bullet (\tau_1,\ \tau_2, \ldots, \tau_p) = (\tau_1,\ \tau_2, \ldots, \tau_p)$, identity exist.

Compatibility: $\forall n, l \in \mathbb{Z}/p\mathbb{Z}$, $n \bullet l \bullet (\tau_1,\ \tau_2, \ldots, \tau_p) = (\tau_{n+l+1},\ \tau_{n+l+2}, \ldots, \tau_p, \tau_1 \ldots, \tau_{n+l}) = (n+l) \bullet (\tau_1,\ \tau_2, \ldots, \tau_p)$

Closure: assume $(\tau_1,\ \tau_2, \ldots, \tau_p) \in X$, thus $\tau_1\tau_2 \ldots \tau_p = e$, then the element after action is $(\tau_2,\ \tau_3, \ldots, \tau_p, \tau_1)$, $\tau_2\tau_3 \ldots \tau_p\tau_1 = {\tau_1}^{-1}(\tau_1\tau_2\tau_3 \ldots \tau_p)\tau_1 = {\tau_1}^{-1}e\tau_1 = e$

It is group action.

Based on the Orbit-Stabilizer Theorem, $\forall x \in X,\ |Orb\,(x)||p$, because p is a prime number, so $|Orb\,(x)|$ only have 2 possible values

$$|Orb\,(x)| = 1 \tag{24}$$

$$|Orb\,(x)| = p \tag{25}$$

When $|Orb\,(x)| = 1$, it can say to every n , $nx = x$, it means $\tau_1 = \tau_2 = \ldots = \tau_p$, so, $\tau^p = e$

Assuming r represents the number of the fixed points. (the number of elements with $|Orb\,(x)| = 1$)

The set X can be divided into some orbits, the orbits ($|Orb\,(x)|$) must be 1 or p .

When $|Orb\,(x)| = 1$, the number of elements is r , when $|Orb\,(x)| = p$,the number of elements is $np,\ n \in \mathbb{Z}$.

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Thus, $|X| = r + np$ can be ontained

$$|X| \equiv r\ (mod\ p) \tag{26}$$

$$|X| = |G|^{p-1} \equiv 0 \left(mod\ p\right) \tag{27}$$

$$r \equiv 0\ (mod\ p) \tag{28}$$

$(e, e, \ldots, e)$ is a fixed point, it is included in r . $r \equiv 0\ (mod\ p)$ and $p \geq 2$, thus $r \geq p$. So, except $(e, e, \ldots, e)$, there must be $p-1$ other fixed point.

Assuming $(\tau, \tau, \ldots, \tau)$ is a fixed point, $\tau \neq e$, based on the definition of fixed point, $\tau^p = e$ can be obtained. Because p is a prime number, and $\tau \neq e$, thus the order of element τ is p .

3.2. Burnside counting theorem

Theorem description:

Assuming G is a finite group, and X is a finite set, let N represents the number of the orbits. $Fig\,(g) = \{x \in Xg \bullet x = x\},\ g \in G$, is the set of fixed points of elements g .

$$N = \frac{1}{|G|} \sum_{g \in G} |Fig\,(g)| \tag{29}$$

Process of proof:

define a set $S = \{(g, x) \in G \times Xg \bullet x = x\}$ is the set of all the fixed points of elements. Then, 2 ways to can be found to calculate $|S|$.

To any elements $g \in G$, the number of fixed points is $|Fig\,(g)|$. Thus

$$|S| = \sum_{g \in G} |Fig\,(g)| \tag{30}$$

To any fixed point $x \in X$, the number of elements is $|Stab(x)|$. Thus

$$|S| = \sum_{x \in X} |Stab(x)| \tag{31}$$

By these two ways, an equation can be obtained.

$$\sum_{g \in G} |Fig\,(g)| = \sum_{x \in X} |Stab(x)| \tag{32}$$

Based on the Orbit-Stabilizer Theorem, it can get an equation

$$|G| = |Orb\,(x)| \bullet |Stab\,(x)| \tag{33}$$

$$|Stab(x)| = \frac{|G|}{|Orb(x)|} \tag{34}$$

Thus

$$\sum_{g \in G} |Fig\,(g)| = \sum_{x \in X} |Stab(x)| = \sum_{x \in X} \frac{|G|}{|Orb(x)|} \tag{35}$$

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Then, the elements in X can divided by the orbits, assume the orbits is $O_1, O_2, \ldots, O_N$, to any point x in the same orbit, $|Orb(x)| = |O_i|$ so every x give the total sum is $\frac{|G|}{|O_i|}$, and every orbit do for the sum is $\frac{|G|}{|O_i|} \times |O_i| = |G|$, and total have N orbits, so it can get the equation

$$N \bullet |G| = \sum_{x \in X} \frac{|G|}{|Orb(x)|} = \sum_{g \in G} |Fig\,(g)| \tag{36}$$

Thus

$$N = \frac{1}{|G|} \sum_{g \in G} |Fig\,(g)| \tag{37}$$

4. Conclusion

In this paper, it studies on the orbit stabilizer theorem in group theory. It used the theory of group action to provide a detailed description and proof of the orbit stability theorem. Next, it gave some more examples, Cauchy theorem and Burnside Counting Theorem, proved them using the orbit stability theorem. From this, it can be seen that the orbit stability theorem has very important applications in many mathematical theories. However, this paper only discusses the basic concepts of the orbit stability theorem and some of its mathematical applications, without making any improvements to it or exploring any interdisciplinary applications. In further discussion, it can study about the application of this theorem in various fields such as physics, or make some improvements.

References

[1] de Lagrange, J. L. (1770). Démonstration d'un théoreme d'arithmétique. Nouveau Mémoire de l'Académie Royale des Sciences de Berlin, 123-133.
[2] Cauchy, A. (1840). Méthode simple et nouvelle pour la détermination complète des sommes alternées formées avec les racines primitives des équations binômes. Journal de mathématiques pures et appliquées, 5, 154-168.
[3] Burnside, W. (1911). Theory of groups of finite order. University.
[4] Bander, M., & Itzykson, C. (1966). Group theory and the hydrogen atom (I). Reviews of modern Physics, 38(2), 330.
[5] Lewin, K., & Gold, M. E. (1999). The dynamics of group action.
[6] Von Cranach, M., Ochsenbein, G., & Valach, L. (1986). The group as a self-active system: Outline of a theory of group action. European Journal of social psychology, 16(3), 193-229.
[7] Shiriaev, A., Perram, J. W., & Canudas-de-Wit, C. (2005). Constructive tool for orbital stabilization of underactuated nonlinear systems: Virtual constraints approach. IEEE Transactions on Automatic Control, 50(8), 1164-1176.
[8] Armstrong, M. A. (1988). Actions, Orbits, and Stabilizers. In Groups and Symmetry (pp. 91-97). New York, NY: Springer New York.