

The Formation of Dark Matter Halo and Photon Decoupling

Kaidong Cao

*Dalian University of Technology and Belarusian State University Joint Institute, Dalian University of Technology, Dalian, China
CaoKaidong@mail.dlut.edu.cn*

Abstract. This paper aims to establish a unified theoretical framework illustrating the joint effect of the evolution of dark matter halo formation and baryon-photon decoupling on the initial conditions of cosmic structure formation during recombination. The event and mechanism during the period $1100 < z < 3400$ are briefly concluded through the analytical modelling of Jeans instability in expanding spacetime, Top-Hat spherical and ellipsoidal collapse simulations, and Boltzmann transport solutions for recombination kinetics. The paper mainly highlights the dark matter virilization dynamics and the mechanism of recombination, emphasizing their roles in establishing the large-scale structure.

Keywords: Dark Matter Halos, Recombination, Photon Decoupling

1. Introduction

Since the 20th century, cosmologists and astronomers have continuously improved our understanding of the origin, composition, and evolution of the universe. Various models have been proposed and tested to explain the formation and evolution of the universe. Based on the observational advances, the Λ CDM model has emerged as the standard framework for describing the large-scale structure and dynamics evolution of the universe in cosmology. One of the primary goal of modern cosmology is to explain the formation and evolution of cosmic structures. In this context, the processes of dark matter halo formation and photon decoupling during the recombination epoch play foundational roles in setting the initial conditions for structure formation, which is irreplaceable.

Dark matter dominates the mass content of the universe. The collapses of the dark matter under the gravity form the halos, providing the gravitational wells into which the baryonic particles can densely accumulate. The paper discuss the dynamics process of halo formation from the linear perturbation growth, nonlinear collapse and virialization. The formation process of halos governs the distribution and properties of cosmic structures.

The recombination and the photon decoupling taking place subsequently mark the transition of the universe from a dense plasma to a neutral, transparent medium. This epoch enables baryonic matter to fall into dark matter potential wells, facilitating the formation of the first galaxies.

The paper aims to establish a unified theoretical framework that connects the evolution of dark matter halos, recombination and photon decoupling. Focusing on the critical redshift range $1100 < z < 3400$, the paper combines the analytical models of Jeans instability in an expanding spacetime,

spherical and ellipsoidal collapse simulations, and Saha, Boltzmann, and Peebles equation of recombination kinetics to illuminate photon-baryon interactions, highlighting their joint influence on the emerging large-scale structure of the universe. By synthesizing the insights and theories from cosmological perturbation theory, gravitational instability and statistical mechanics, this work aims to provide a coherent picture of how initial density fluctuations evolved into the cosmic structures observed today.

2. The formation of the dark matter halo

Most of the galaxies observed are formed in the dark matter halos since they provide the condition of coalescence of the baryonic particles with the potential well. As one of the most basic and fundamental components in large-scale structure the of the universe and the birthplace of the most galaxies, the study of the dark halos and their formation can help us deepen and widen our comprehension and perspective of galaxy evolution and the model of the universe, which is of the vital importance to shape the stellar structure.

2.1. Jeans instability

In order to describe the universe with gravity, a non-relativistic homogeneous fluid without viscosity and rotation is considered. For simplicity, the most properties of the fluid can be generally characterized by four equations. Firstly, considering the most basic property of the fluid, the conservation law of the mass, the first introduced equation is the continuity equation, which is,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \quad (1)$$

where ρ and $\vec{u}$ is the density and velocity of the fluid respectively,

The second is the Euler equation representing the conservation of the momentum of the fluid. In fact, the Euler equation is the differential equation describing the motion of the in-viscid fluid basing on the application of Newton's second law [1] $F = ma$ for a continuous system,

$$\rho \frac{d\vec{u}}{dt} = \rho \left(\frac{\partial}{\partial t} + \vec{u} \cdot \nabla \right) \vec{u} = -\nabla P - \rho \nabla \Phi \quad (2)$$

where Φ is the term of gravitational potential, which can be determined and expressed by the Poisson equation,

$$\nabla^2 \Phi = 4\pi G \rho \quad (3)$$

The last one is the equation of state,

$$P = P(\rho, T) \quad (4)$$

For simplicity, consider a barotropic fluid [2], relating P to ρ ,

$$\frac{\partial P}{\partial \rho} \equiv c_s^2 \quad (5)$$

where c_s is defined as the sound speed of the fluid.

There are also small perturbations of pressure, density and gravitational potential evolves in time,

$$P(t, \vec{x}) = \bar{P} + \delta P(t, \vec{x}) \quad (6)$$

$$\rho(t, \vec{x}) = \bar{\rho} + \delta \rho(t, \vec{x}) \quad (7)$$

$$\Phi(t, \vec{x}) = \bar{\Phi} + \delta \Phi(t, \vec{x}) \quad (8)$$

As long as all perturbations are small, we can linearize the fluid equations, dropping all terms that are higher than first order in perturbations [2]. The equation (1.1) (1.2) (1.3) become

$$\frac{\partial \delta \rho}{\partial t} + \nabla \cdot \left(\bar{\rho} \vec{u} \right) = 0 \quad (9)$$

$$\bar{\rho} \frac{d\vec{u}}{dt} = -\nabla \delta P - \bar{\rho} \nabla \delta \Phi \quad (10)$$

$$\nabla^2 \delta \Phi = 4\pi G \delta \rho \quad (11)$$

The equations can be combined into one

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2 \nabla^2 - 4\pi G \bar{\rho} \right) \delta \rho = 0 \quad (12)$$

With the ansatz [1]

$$\delta\rho = C(\vec{k}) \exp\left(i\left(\omega t - \vec{k} \bullet \vec{x}\right)\right) \quad (13)$$

Where $\omega^2 = c_s^2 k^2 - 4\pi G \bar{\rho} = c_s^2 (k^2 - k_J^2)$, c_s indicates the ability of pressure.

Define $k_J = \sqrt{\frac{4\pi G \bar{\rho}}{c_s^2}}$ as the Jeans wavenumber, by $\lambda_J = \frac{2\pi}{k_J}$ and $M_J = \frac{4\pi}{3} \left(\frac{\lambda_J}{2}\right)^3 \bar{\rho}$ the Jeans wavelength and Jeans mass is

$$\lambda_J = \sqrt{\frac{\pi c_s^2}{G \bar{\rho}}} \quad (14)$$

$$M_J = \frac{\pi^{\frac{5}{2}} c_s^3}{6 G^{\frac{3}{2}} \bar{\rho}^{\frac{1}{2}}} \quad (15)$$

The Jeans length indicates the minimum spatial scale of perturbation growth. Any perturbation with a larger size than the Jeans length will typically grow exponentially quickly due to the effect of gravity [3]. The exponential growth of the fluctuations is called the Jeans instability [2]. The Jeans mass indicates the minimum mass to overcome the pressure support and trigger gravitational collapse.

Jeans instability provides a theoretical basis for the early stage of the formation of dark matter halos, revealing the primary mechanism of the growth of the dark matter halo.

Unfortunately, if an expanding universe is considered, Jeans's theory (1.12) is not applicable to the formation of galaxies and dark matter halos any more, because in Jeans' assumption the fluid exist as a static medium [3] without considering the expansion described by Hubble's law. To describe the expanding universe the Hubble's law should be considered, since the scale factor $a(t)$ has a great effect on the suppression of the perturbation.

$$\vec{r}(t) = a(t) \vec{x}(t) \quad (16)$$

$$\vec{u} = \dot{\vec{r}} = H \vec{r} + v \quad (17)$$

Combining with the equations (1.9) (1.10) and (1.11), the equation (1.12) in the expanding universe turns out to be

$$\ddot{\delta} + 2H\dot{\delta} - c_s^2 \left(\frac{1}{a^2} \nabla^2 + k_J^2 \right) \delta = 0 \quad (18)$$

Where $\delta = \frac{\delta P}{P} = \frac{\delta \rho}{\rho}$ and $k_J = \sqrt{\frac{4\pi G \bar{\rho}}{c_s^2}}$

Then the relationship between the linear perturbation of different component in different era and time can be obtained by combing the equation (1.18) and the Friedman equation.

2.2. Top-hat spherical collapse

As the size of the disturbance grows to $\delta \approx 1.06$ [2], the linear perturbation theory mentioned above is not applicable any more, the growth of the dark matter halo is apart from the expansion described by the Hubble law. The nonlinear collapse model needs to be introduced. The simplest model to describe the nonlinear growth of structures under the action of gravity is the sphere collapse model [4].

For convenience, consider a universe consisting only dark matter and dark energy with zero flatness ($k = 0$) and homogeneity. The Friedman equation of the universe above can be written as [5,6]

$$H^2 \equiv \left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3} = \frac{8\pi G}{3} \left(\bar{\rho}_m + \bar{\rho}_{de} \right) \quad (19)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left(\rho + \frac{3p}{c^2} \right) + \frac{\Lambda c^2}{3} = -\frac{4\pi G}{3} \left[\bar{\rho}_m + \left(1 + 3\omega \right) \bar{\rho}_{de} \right] \quad (20)$$

Where $k = 0$, $\omega \equiv \frac{P}{\rho}$ and for the dark energy $\omega = -1$

The Spherical Collapse model considers an isolated, symmetric spherical over-dense region with a top-hat density profile and uniform density $\rho(t) = \bar{\rho}(t) + \delta\rho(t)$, immersed in a homogeneous universe with energy density $\bar{\rho}(t) = 0$ [5]. Inside the local sphere, exist the scale factor r

$$\frac{\ddot{r}}{r} = -\frac{4\pi G}{3} \left(\rho_m + \rho_{de} + 3p_{de} \right) = -\frac{4\pi G}{3} \left(\frac{M}{\frac{4\pi r^3}{3}} - 2\rho_{de} \right) \quad (21)$$

$$\ddot{r} = -\frac{GM}{r^2} + \frac{8\pi G}{3} \rho_{de} r \quad (22)$$

When $\rho_{de} = 0$ (Einstein-de Sitter universe $k = 0$ (flat universe) and $\Lambda = 0$), the solution becomes periodical.

$$\ddot{r} = -\frac{GM}{r^2} \quad (23)$$

which can be written as,

$$\frac{1}{2} \dot{r}^2 - \frac{GM}{r} = E \quad (24)$$

,where E is a constant and $E < 0$ [2].

The solution can be represented as

$$r(\theta) = -\frac{GM}{2E} (1 - \cos\theta) \quad (25)$$

$$t(\theta) = \frac{GM}{(-2E)^{\frac{3}{2}}} (\theta - \sin\theta) \quad (26)$$

Initially, r increases due to the Hubble expansion, when $t = t_a$ ($\theta = \pi$), the turn-over time, r decrease, and finally $r = 0$ when $t = 2t_a$ ($\theta = 2\pi$), which is called the collapse. In the Einstein-de Sitter universe, considering the linear perturbation regime, the critical perturbation is around 1.69 which means, if density contrast reaches 1.69 in the Top-Hat Spherical Collapse model it should be interpreted a complete collapse [1]. If $\rho_{de} \neq 0$, there is no closed-form solution for the equation, however, by simulation, the effect of the dark energy is minor in the high red-shift region.

2.3. Ellipsoidal collapse

The Spherical Collapse model is an over idealized model for explaining the early stage of the formation of the dark matter halo. The actual distribution of cosmic matter is not perfectly spherically symmetric. Ellipsoidal Collapse is proposed supposing the initial density perturbations are ellipsoidal in shape with three different principal axis. When the dark matter collapses under gravity, the collapse speed and timing along different principal axes are different. The equations of motion for each axis include both the self-gravitational term and the tidal coupling term.

There are two main factors affecting the evolution of the halo, the shear tensor (Σ) and angular momentum (L). The Shear tensor caused by the anisotropic distribution of density field and collapse speed along the axis tend to accelerate the collapse, while the angular momentum caused by tidal torque tend to delay down the collapse of the halo.

Suppose that the rotational velocity has a unique component only on the z-axis and there is no component for the initial vorticity (acquired after shell crossing) on the x-axis and y-axis [7]. Since the rotational velocity is along the z-axis, centrifugal force only acts on the x-y plane, there is no angular momentum term for the evolution equation of the radius on the z-axis. Therefore, the z-axis

radius of the ellipsoid collapses first, while the x and y-axis radii are supported by angular momentum.

The kinetic equation can be described as [7],

$$\frac{\ddot{\rho}}{\rho} - \frac{4}{3} \left(\frac{\dot{\rho}}{\rho} \right)^2 - 4\pi G\rho - \Sigma^2 + \frac{8L^2}{a_1 a_2 (a_1 + a_2)^2} = 0 \quad (27)$$

where a representing the semi-axis of the ellipsoid.

2.4. Virilization and NFW profile

Theoretically, when the density of the singularity point is infinite, it would become a black hole. However, the random motion and interactions among the matter lead to the equilibrium. The kinetic energy and potential energy are finally balanced in the area, forming a dark matter halo. The process is called Virilization since the kinetic energy and potential energy follow the Virial theorem $\overline{T} = -\frac{1}{2} \overline{V}$ at last. During the collapse, the decreased average potential energy $\overline{V}$ is converted into the average kinetic energy $\overline{T}$ due to the conservation law of the energy against the collapse at the same time and finally reaches the equilibrium, forming a dark matter halo of which the density is roughly 178 times greater than its background critical density of Spherical Collapse in the Einstein-de Sitter universe [1].

In the case of the Ellipsoidal Collapse model, angular momentum balance shear tensor. Angular momentum competes with the shear tensor delaying the expected time to reach the Virilization, and this process is called Previrilization. The density of the halo contrast at virilization δ_v is about 60, which is reduced with respect to the expected value in the Spherical Collapse model ($\delta_v \approx 178$) [7]. That is, the non-spherical shape and tidal interactions impede the collapse.

Additionally, the Virilization of the dark matter halo is different from the baryonic particles due to the collision-less properties of dark matter particles and the absence of the electromagnetic interaction. The advanced studies of the Virilization of the dark matter halo are trying to explain it together with the Phase-Space Mixing. By applying the Ensemble and Liouville's theorem, the initial localized dense distribution gradually spreads and mixes with the surroundings uniformly.

The density distribution of the dark matter halos follows the classical empirical NFW profile, which was first proposed by F. Navarro, S. Frenk and M. White in 1996.

The equilibrium density profiles of CDM halos of all masses can be accurately fitted over two decades in radius by the simple formula [8]

$$\frac{\rho(r)}{\rho_{\text{crit}}} = \frac{\delta_c}{\left(\frac{r}{r_s} \right) \left(1 + \frac{r}{r_s} \right)^2} \quad (28)$$

where r_s is the scale radius, δ_c is the characteristic density (which is dimensionless), and $\rho_{\text{crit}} = \frac{3H^2}{8\pi G}$ is the critical density for closure.

The dark matter distribution predicted by the NFW Profile fits well with the rotation curve observations of most spiral galaxies, however, there are still some problems with NFW Profile need to be addressed, for example, the core/cusp problem in the field of low surface brightness disk and gas-rich dwarf galaxies [9]. To address these problems several hypotheses have been mentioned, the most famous one was the Warm Dark Matter [10]. Compared with CDM, the Warm Dark Matter with higher velocity and lower mass can decrease the halo concentrations and core densities, preventing the sudden increase of the dark matter halo in density at its center and resolving ambiguities in the NFW profile's applicability to low-mass halos.

3. The interaction between dark matter and baryons

The primary interaction between dark matter and baryons is the gravitational force. The potential well of a dark matter halo provides the initial attraction for baryonic gas to fall in. As the gas collapses, it becomes thermalized. To continue condensing and form stars, the gas has to radiate the thermal energy caused by the gas pressure away. For baryonic gas, radiative cooling, primarily through atomic transitions, is efficient in the dense inner regions of halos to radiate thermal energy. However, since dark matter does not participate in electromagnetic interactions, its hot pressure cannot be radiated away effectively; dark matter thus forms only a sparse halo rather than a dense one, unlike baryons. The gravitational wells created by dark matter lead to the accumulation of sufficient baryonic mass to form galaxies within the age of the universe. While dark matter initiates the process, baryons also affect the structure of the dark matter halo itself.

As baryons cool and condense into a very dense central region, they deepen the gravitational potential. Dark matter particles, responding to this changing potential, gain energy and move in tighter orbits to maintain their angular momentum, leading to further contraction of the dark matter halo and an increase in its central density.

Conversely, violent energetic feedback from baryonic processes, such as powerful supernova explosions, can transfer energy to dark matter particles, flattening the halo's central density cusp into a shallower core.

4. Recombination and photon decoupling

Though the linear perturbation of the dark matter has already grown when $z \approx 3400$, the first micro dark matter halo has formed when $z \approx 30$. During that period, the Recombination and photon decoupling have already taken place.

The universe was composed of the plasma mainly made up of electrons, photons and the baryons such as, protons, neutrons, helium nuclei and other heavier nuclei when $z > 1100$. Photons and electrons exchange their energy by the Compton scattering, maintaining the photon-baryon plasma and the thermal equilibrium state. However, as the expansion of the universe, the Compton scattering becomes not efficient and strong enough to keep the strong coupling between the photons and the baryons, the next important stage in the universe takes place, which is called decoupling. The decoupling epoch happens as the temperature drops below 1 eV [6], since then the quantity of the free electrons in the universe drops immensely.

4.1. Photon-baryon coupling and baryonic acoustic oscillations

In the beginning of the matter-dominated era, around $1100 < z < 3400$, the electrons, photons and baryons, mainly protons, were in the state of thermal equilibrium as plasma, which is called the

Photon-Baryon Coupling.

The radiation pressure was transmitted by the photons to suppress the growth of the baryonic matter density perturbation and the total fluid propagated energy through the oscillation of sound waves. The sound speed of the photon-baryon fluid was about $\frac{c}{\sqrt{3}}$, which was much greater than that of the dark matter.

The baryonic matter density perturbation was held by the photons. The photons and baryonic matters oscillated together in the local area. The sound waves caused by the oscillation in the dense fluid composed of the photons and baryonic matters are known as the baryonic acoustic oscillations [1], the BAO.

4.2. Recombination

When electrons started to combine with nuclei. The decoupling of photons got greatly promoted since the interaction between electrons and photons decreased, while the interaction between electrons and nuclei increased. The main reaction is



4.3. Saha equation

In the Canonical Ensemble in Statistical Mechanics, the chemical potential of the particles participant in the ionization can be simplified as [11],

$$\mu = \left. \frac{\partial F}{\partial N} \right|_{T,V} = -k_B T \ln Z = -k_B T \ln \left\{ \frac{g_s}{n_s} \left(\frac{2\pi m_s k_B T}{h^2} \right)^{\frac{3}{2}} \right\} + m_s c^2 \quad (30)$$

the Coulomb interaction between the particles is ignored.

Considering the chemical equilibrium, $\mu_e + \mu_p = \mu_H + \mu_\gamma$ in the Recombination. Since $\mu_\gamma = 0$ [2], the equilibrium is able to be written as

$$\frac{n_e n_p}{n_H} = \frac{g_e g_p}{g_H} \cdot \left(\frac{2\pi m_e m_p k_B T}{m_H h^2} \right)^{\frac{3}{2}} \cdot e^{-\frac{(m_e + m_p - m_H)c^2}{k_B T}} \quad (31)$$

Define the free electron fraction of recombination $X_e \equiv \frac{n_e}{n_e + n_H} = \frac{n_p}{n_p + n_H}$, the Ionization Energy $E_I = (m_e + m_p - m_H)c^2$, $n_b = n_p + n_H$ and $\eta = \frac{n_b}{n_\gamma}$. Since $m_p \approx m_H$, $n_p = n_e$, $g_e = g_p = 2$ and $g_H = 4$ [12], therefore,

$$\frac{X_e^2}{1 - X_e} \eta n_\gamma = \left(\frac{2\pi m_e k_B T}{h^2} \right)^{\frac{3}{2}} \bullet e^{-\frac{E_I}{k_B T}} \quad (32)$$

By the Planck's Black-body radiation law, $n_\gamma = 16\pi\zeta(3) \left(\frac{k_B T}{hc} \right)^3$ [13], thus [2],

$$\frac{1 - X_e}{X_e^2} = \frac{2\zeta(3)}{\pi^2} \eta \left(\frac{2\pi k_B T}{m_e c^2} \right)^{\frac{3}{2}} \bullet e^{\frac{E_I}{k_B T}} \quad (33)$$

This is called the Saha equation. Applying the equation of temperature $T = T_0(1 + z)$, the relationship between electron fraction and redshift is

$$\frac{1 - X_e}{X_e^2} = \frac{2\zeta(3)}{\pi^2} \eta \left(\frac{2\pi k_B T_0(1 + z)}{m_e c^2} \right)^{\frac{3}{2}} \bullet e^{\frac{E_I}{k_B T_0(1+z)}} \quad (34)$$

,where $T_0 = 2.73K$, $E_I = 13.6eV$, $\zeta(3) \approx 1.20$ and $\eta = 6 \times 10^{-10}$.

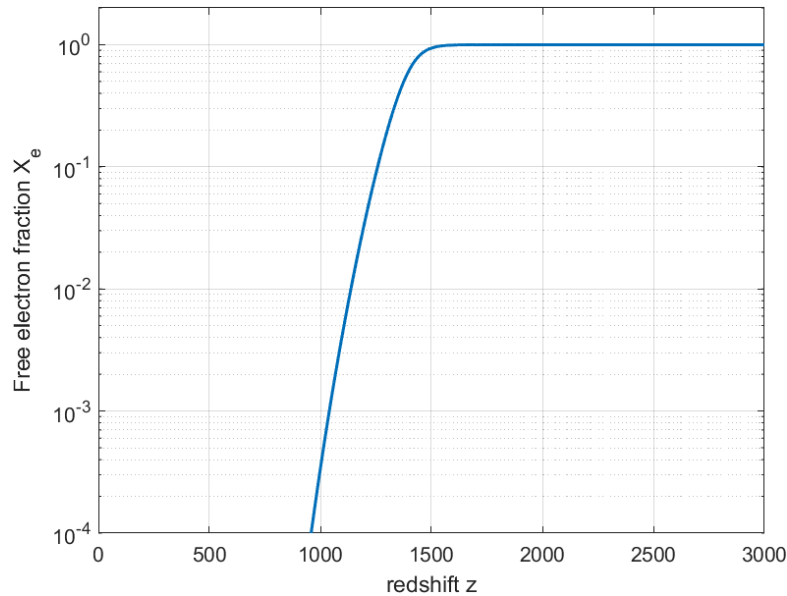


Figure 1. The relationship between the free electron fraction and redshift as described by the Saha equation

4.4. Boltzmann equation

For the purpose of describing the statistical behavior and evolution of the distribution of the micro particles in a thermodynamic system which is in a non-thermodynamic equilibrium state, the Boltzmann equation is often applied.

Initially, considering that each particle state in the system can be described by its position and momentum in classical physics at time t , the number of particles around the phase-space $(\mathbf{x}_i, \mathbf{p}_i)$ is

$$N(\mathbf{x}_i, \mathbf{p}_i, t) = f(\mathbf{x}_i, \mathbf{p}_i, t) \frac{d^3x \bullet d^3p}{h^3} \quad (35)$$

,where $f(\mathbf{x}, \mathbf{p}, t)$ is the distribution function .

Based on the collision-less assumption (the number of particles around phase-space $(\mathbf{x}_i, \mathbf{p}_i)$ is conserved), the total time derivative of $f(\mathbf{x}, \mathbf{p}, t)$ is zero.

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{\mathbf{x}} \bullet \nabla_{\mathbf{x}} f + \dot{\mathbf{p}} \bullet \nabla_{\mathbf{p}} f = 0 \quad (36)$$

To account for collisions, a collision term should be introduced to describe the movement of particles among different phase-spaces due to particle interactions.

$$\frac{\partial f}{\partial t} + \dot{\mathbf{x}} \bullet \nabla_{\mathbf{x}} f + \dot{\mathbf{p}} \bullet \nabla_{\mathbf{p}} f = C[f] \quad (37)$$

By considering the expanding universe and the isotropic assumption of the universe,

$$\frac{\partial f}{\partial t} - H\mathbf{p} \bullet \nabla_{\mathbf{p}} f = C[f] \quad (38)$$

Integrate both sides of the equation and account for the collision circumstance of $1 + 2 \leftrightarrow 3 + 4$ [6]. Taking the evolution equation of n_1 as an example, the Boltzmann equation of the classical dilute gas follows the Boltzmann distribution can be written as [14],

$$a^{-3} \frac{d(n_1 a^3)}{dt} = n_1^{(0)} n_2^{(0)} \langle \sigma \nu \rangle \left\{ \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right\} \quad (39)$$

where $\langle \sigma \nu \rangle$ is the scattering cross section of the particles, $n_s \equiv g_s \int \frac{d^3p}{h^3} f(\mathbf{x}, \mathbf{p}, t) = g_s e^{\frac{\mu_s}{T}} \int \frac{d^3p}{h^3} e^{\frac{-E_s}{T}}$ is the number density under Boltzmann distribution and $n_i^{(0)}$ is the number density when the chemical potential of the particle is 0.

The relationship between electron fraction and redshift for Recombination $e^- + p \leftrightarrow {}^1_1H + \gamma$ is [2]

$$-\frac{dX_e}{dz} \bullet H(z) \bullet (1+z) = \left(1 - X_e\right) \left(\frac{m_e c^2 k_B T}{2\pi}\right)^{\frac{3}{2}} \bullet e^{-\frac{E_I}{k_B T}} \langle \sigma \nu \rangle - X_e^2 \eta m_\gamma \langle \sigma \nu \rangle \quad (40)$$

While $\langle \sigma \nu \rangle = 9.78 \frac{\alpha^2}{m_e^2 c^4} \left(\frac{E_I}{k_B T}\right)^{\frac{1}{2}} \ln\left(\frac{E_I}{k_B T}\right)$, $\alpha = \frac{1}{137}$, and $H^2(z) = H_0^2 [\Omega_m(1+z)^3 + \Omega_r(1+z)^4 + \Omega_\Lambda]$. In the case of Recombination, $\Omega_r = \Omega_\Lambda = 0$ and $\Omega_m = 1$ and $H_0 \approx 70 \text{ km} \bullet \text{s}^{-1} \bullet \text{Mpc}^{-1} \approx 2.27 \times 10^{18} \text{ s}^{-1}$.

For simplicity, (3.12) can be expressed as

$$-\frac{dX_e}{dz} \bullet H(z) \bullet (1+z) = \left[\beta(T) (1 - X_e) - \alpha(T) X_e^2 \eta m_\gamma \right] \quad (41)$$

,where $\alpha(T) \equiv \langle \sigma \nu \rangle$ and $\beta(T) \equiv \langle \sigma \nu \rangle \bullet \left(\frac{m_e c^2 k_B T}{2\pi}\right)^{\frac{3}{2}} \bullet e^{-\frac{E_I}{k_B T}}$.

As the universal equation for describing the distribution of particles, the Boltzmann equation can be applied in various circumstances. It should be noted that, for simplicity, Equation (3.12) only considers the main reaction $e^- + p \leftrightarrow \frac{1}{2}H + \gamma$ in recombination, without taking the state transitions of hydrogen atoms into account.

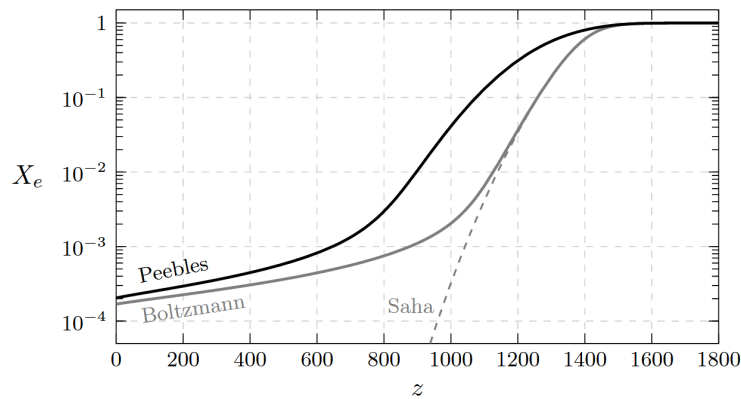


Figure 2. The comparison of the relationships between the free electron fraction and redshift as described by the Saha equation, the Boltzmann equation, and the Peebles equation [2]

4.5. Peebles equation

There are also many other equations proposed based on different hypotheses to describe the electron fraction, without combining the Boltzmann equation to describe the remaining reactions in the recombination. The Peebles equation considers the distribution of excited states of hydrogen and the recombination rate [14].

To describe the electron fraction, Peebles Equation introduces the three-level system of the hydrogen atom, including the continuous state (free electrons and protons), the ground state $n = 1$ and the first excited state $n = 2$ ($2s$ and $2p$) .

Peebles pointed out that the captures to excited states of the hydrogen atom only contribute meaningfully to recombination when the first excited state $n = 2$ ($2s$) of the hydrogen atom decays into two photons, or when the expansion of the universe red-shifts the Lyman alpha photon (from $n = 2$ ($2p$) to $n = 1$) to the extent that it can no longer excite a ground state atom [6], so-called the resonance escape. To account for these circumstances, Peebles introduced a correction factor $C_r(T)$.

$$C_r(T) \equiv \frac{\Lambda_{2\gamma} + \Lambda_\alpha}{\Lambda_{2\gamma} + \Lambda_\alpha + \beta_\alpha} \quad (42)$$

,where $\Lambda_{2\gamma} = 8.227 \text{sec}^{-1}$ represents the rate of two-photon decay,
 $\Lambda_\alpha = \frac{27}{1024 \pi^3 \zeta(3)} \frac{H(z) \cdot (E_I h c)^3}{(1 - X_e) \eta (k_B T)^3}$ represents the rate of recombination via resonance escape and
 $\beta_\alpha = \beta \cdot e^{\frac{3E_I}{4k_B T}}$. represents the ionization rate of the first excited state.

Multiply it to the right-hand side of the Boltzmann equation (3.13).

$$-\frac{dX_e}{dz} \cdot H(z) \cdot (1 + z) = C_r(T) \left[\beta(T) (1 - X_e) - \alpha(T) X_e^2 \eta n_\gamma \right] \quad (43)$$

,where $\alpha(T) \equiv \langle \sigma \nu \rangle$, $\beta(T) \equiv \langle \sigma \nu \rangle \cdot \left(\frac{m_e c^2 k_B T}{2\pi} \right)^{\frac{3}{2}} \cdot e^{-\frac{E_I}{k_B T}}$.

4.6. Photon decoupling and CMB

The Recombination of electrons and protons reduce the interaction between electrons and photons, which facilitates the decoupling of photons.

The main interaction between the photons and electrons was the Compton scattering to exchange their energy and momentum.

$$e^- + \gamma \leftrightarrow e^- + \gamma \quad (44)$$

After the Photon Decoupling, the photon freely travels through the universe. The universe becomes “transparent”.

The freely travelled photons became the first free photons. From then on, CMB (cosmic microwave background) photons began to freely propagate in the universe.

The radiation during the decoupling period became the CMB, which is one of the key pieces of evidence for the Big Bang theory, inflation and Λ CDM. All the effective and qualified cosmic model proposed now should own the ability to explain the CMB. The uniform temperature of CMB with small perturbation reflects the homogeneity of the early universe and the perturbation of CMB also became one of the most useful and effective tools to study the BAO.

Recently, the Blackbody Radiation Inversion (BRI) method with the COBE/FIRAS dataset was also applied in calculating the monopole and dipole temperatures and their uncertainties of CMB. The dipole spreading exceeds the prediction from relative motion, which indicates the additional contributions from blackbody mixing [15].

5. Conclusion

The formation of the dark matter halos is one of the most vital and necessary events in the evolution of the universe. Dark matter halos are closely related to the large-scale structure in the universe. Since the beginning of the matter-dominant era, the density perturbations of dark matter grew linearly and non-linearly collapsed into the halos due to virialization. NFW profile shows the density distribution of the halo, several models have also been proposed to address the problem the NFW profile have encountered.

The recombination and photon decoupling allow coalescence of the baryons into the potential wells formed by the dark matter, making great effect on the large-scale structure. The Boltzmann equation well describes the fraction of the free electrons during the recombination. The CMB also preserves the strong evidence of the Big Bang theory and inflation.

Future observations of CMBs and galaxies will deepen our comprehension of the formation of the dark matter halos and photon decoupling, and broaden our horizontal field of view. By combining observations with advanced models, our understanding of cosmology and astronomy will be greatly improved.

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