

# *The Application and Proof of Group Theory in the Three-body Problem*

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**Abstract.** There are various symmetries of great importance in modern mathematical physics. These symmetries can enormously reduce computation and simplify the solution of many complicated dynamic systems and differential equations. Among the applications of the group theory, the applications to solving dynamic equations of the planar three-body problem is a study worthy case. The aim of this paper is to illustrate the application of group action to the three-body problem. The author introduced the work of Chenciner and Montgomery published on the Annual of Mathematics in 2000. In this paper, the author illustrates the proof sketch of the main Lemma in the paper of A.C. and R.M. and analyzes the main steps concerning group action. Two applications of group were demonstrated in this paper. The first one is reducing the solution space of the problem with the symmetry group of the sphere,  $SO(2)$ . The second one is the action of the Klein group, used to generate the entire orbit of the solution from a piece of loop. Finally, the potential applications of group theory to more dynamical problems.

**Keywords:** Group theory, three-body problem, group action, periodic solution

## **1. Introduction**

The idea of group has a rather long history which can date back to work of Lagrange in the late 18th century. The work of Cauchy and Galois is commonly regarded as the beginning of group theory [1-4]. The study about the groups has brought fruitful results to the study of a variety of fields in mathematics and modern physics. The concepts of the Galilean group, the Lorentz group and the Poincaré group are of great importance when understanding modern physics. The method of group action and finite groups also played an important role in the study of mathematical physics and dynamical systems.

Three-body problem is an ancient problem first formulated and studied by Newton in his epoch-making work Principia, considering the interaction of three bodies through Newtonian gravity without any extra restriction [5]. In the late 19th century, the work of Poincaré fundamentally changed the research method for the three-body problem and introduced and proved that there cannot be an analytic solution to the motion equation [6,7]. The problem had attracted the interest of the greatest mathematicians for 300 centuries, besides Newton and Poincaré; Euler and Lagrange had separately given solutions to the problems by considering the symmetry in 1767 [8] and 1772 [1].

Among all the potential solutions, the periodic and stable ones are cared for most by mathematicians and physicists, due to the celestial mechanical background of the three-body problem. Whether there can be a stable periodic solution to the problem remains unknown. In recent years, a series of periodic solutions were given by using the group methods and considering the symmetry situation [9-12]. By considering the symmetry of the cyclic groups there appears a family of solutions of the equation [9,10]. The only known solution with Kolmogorov-Arnold-Moser stability is found by Moore and proved by Alain Chenciner and Richard Montgomery [11,12]. In the proof of Chenciner and Montgomery, they considered the symmetry of  $\mathbb{R}/T\mathbb{Z}$  and  $\mathbb{R}^2$ , where  $T$  is an arbitrary real number. They proved the existence of the stable orbit by defining an action of the Klein group on them [12]. There are some new approaches based on the work of Chenciner and Montgomery, based on the idea of reducing the problem by symmetry and group action [13-15]. One important work is the proof of Marchal's Conjecture [13].

In this paper, the main ideas of the group action method and the main steps using the group action are introduced. The author demonstrates the way symmetry works in the approach to the figure-eight shaped orbit and shows other possible applications of the method of group action in dynamics.

## 2. Definition and problem statement

### 2.1. Basic concepts

Solving the dynamic equation of the motion for three bodies under Newton's laws is the three-body problem. The equation of motion can be written as follows:

$$\begin{cases} \ddot{x}_1 = \frac{m_2}{|x_1 - x_2|^3} (x_1 - x_2) + \frac{m_3}{|x_1 - x_3|^3} (x_1 - x_3) \\ \ddot{x}_2 = \frac{m_1}{|x_2 - x_1|^3} (x_2 - x_1) + \frac{m_3}{|x_2 - x_3|^3} (x_2 - x_3) \\ \ddot{x}_3 = \frac{m_2}{|x_3 - x_2|^3} (x_3 - x_2) + \frac{m_1}{|x_3 - x_1|^3} (x_3 - x_1) \end{cases} \quad (1)$$

In the equation (1),  $\ddot{x}_i$  represent the acceleration of each body and the right-hand sides represent the gravitational force from the other two bodies. The variables  $x$  are position vectors in a three-dimensional Euclidean space representing the position of the three bodies in the space.

Since the gravity field constructed by the three bodies is a potential field, the dynamic equation can be reduced as formula (2), with the definition of potential energy  $U$ :

$$m_i \ddot{x}_i = \frac{\partial U}{\partial x_i}, \quad U = - \left( \sum_{1 \leq i, j \leq 3} \frac{m_i m_j}{\|x_i - x_j\|} \right) \quad (2)$$

In equation,  $m_i$  denotes the mass of the  $i$ th body, and  $x_i$  denotes the coordinate of the  $i$ th body.

Finding solutions to the three-body problem in three-dimensional space is beyond the capability of human mathematics. Therefore, most known solutions, especially the periodic solutions are concerning a restricted initial condition of the three bodies. As the celestial fact of the solar system, the restriction assumes the three bodies move in a two-dimensional plane, which is equivalent to the statement that in the initial condition, the speed of the three bodies is on the same plane. The three-body problem with such restriction is called the planar three-body problem. The first solution for the problem was given by Euler considering the planar case [8]. The research on the planar cases of

Lagrange aroused the method considering the symmetry of the system. The work of Chenciner and Montgomery also considers the planar case in which the three bodies have equal mass.

The solutions considered in this paper are periodic, which means after a fixed period of time the positions of the three bodies will return to the initial position. A classic example of a periodic solution for an equation is the solution of the vibration of simple pendulum  $\theta(t) = A \sin(\sqrt{\frac{L}{g}}t + \phi)$ , in which  $A$  denotes the angular amplitude,  $L$  is the length of the pendulum,  $g$  is the gravitational acceleration, and  $\phi$  denotes the initial angle of the pendulum. The solution of the equation is the function of time which represents the angle of the pendulum in each time. Since  $\sin(x + \phi)$  is a periodic function, it is easy to see that the angle will return to the initial angle after each period, thus this is a periodic solution of the equation. The existence and primary theory of the periodic solution for three-body problem was first proven by Poincaré in 1890 [6].

Given the initial position of the three-body, all the points of the vectors in arbitrary time construct the orbit of a solution. If the solution is periodic, the orbits of the solution is always a finite number of closed curves. The centre of the paper focuses on some method of group theory used to find special periodic and stable orbits.

## 2.2. Statement of main theorem

A periodic solution for the three-body problem was provided by Chenciner and Montgomery with group action in 2000 [12]. The following part is going to give the basic definition and statement of the main theorem of their result. The paper considered three bodies of unit mass in the two-dimensional plane.

Before illustrating the main theorem, the definition of actions of the Klein group used in the following paper have to be given first. They were group actions from  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  to  $\mathbb{R}/T\mathbb{Z}$  and to  $\mathbb{R}^2$  respectively.

Definition 2.2.1: Let  $\sigma$  and  $\tau$  be two generators of the Klein group, the action to the set  $\mathbb{R}/T\mathbb{Z}$  is defined as follow:

For any element  $a$  of the set,  $\sigma \bullet a = a + \frac{T}{2}$ ,  $\tau \bullet a = -a + \frac{T}{2}$ .

Definition 2.2.2: Let  $\sigma$  and  $\tau$  be two generators of the Klein group, the action to the set  $\mathbb{R}^2$  is defined as follows:

For any element  $(a, b)$  of the set,  $\sigma \bullet (a, b) = (-a, b)$ ,  $\tau \bullet (a, b) = (a, -b)$ .

Theorem 2.2.1: An “eight”-shaped orbit of the motion of the three bodies can be constructed. Denote the orbit as loop  $p : (\mathbb{R}/T\mathbb{Z}), 0 \rightarrow \mathbb{R}^2, 0$  where  $T$  is the period of the motion. The loop has properties (1)-(3):

(1) For any  $a$ , the loop  $p$  satisfies:  $p(a) + p(a + \frac{T}{3}) + p(a + \frac{2T}{3}) = 0$ .

(2)  $p$  is commutative with  $\sigma$  and  $\tau$  under the composition of mappings:  $p(\sigma \bullet a) = \sigma \bullet p(a)$ ,  $p(\tau \bullet a) = \tau \bullet p(a)$ .

(3) The loop  $x : (\mathbb{R}/T\mathbb{Z}) \rightarrow \mathbb{R}^6$  defined by:  $x(t) = (p(a + \frac{2T}{3}), p(a + \frac{T}{3}), p(a))$ . Where  $x(t)$  is a periodic solution to the problem.

Though the whole proof of the theorem concerning variational method and analysis is definitely worthy of demonstrating, this paper is going to focus only on the steps concerning the group theory. The next part of this paper is going to show how to use group action to construct the whole solution  $x(t)$  for the planar three body problems.

### 3. Proof of the main theorem

In this part the author will go through the sketch of the proof of the main theorem and give explanations in detail about the steps concerning the group action of the Klein group. In the main part of the first half of the paper [12], Alain Chenciner and Richard Montgomery used the variational method and excluded the potential collision in the motion of the three bodies and proved the existence of a solution between two specific configurations of the three bodies denoted by  $E_3$  and  $M_1$ . In the second half of their paper, they used the group action to prove the existence of the whole eight-shaped orbit.

Chenciner and Montgomery first used the symmetry of  $SO(2)$  to project the image in  $\mathbb{C}^3$  in to  $\mathbb{R}^3$  so that the symmetry of the Klein group can be used to generate the whole orbit with a given piece of solution, thus give out the closed orbit required. After that, some well-known properties about central configurations [14], were introduced. The configurations were considered as points on a sphere in the reduced space. The sphere represents all the oriented triangles with inertia 1. Obviously, if take the three bodies as the points and the direction of the speed as the orientation, all the configurations without collision of the three body were on the sphere. This sphere was called shape sphere. There are some configurations studied by Euler [8] that have made great contribution to the construction of the solution.

Briefly, the construction of the orbit was inspired by two kinds of well-known configuration, the Euler configuration and the Isosceles configuration. The  $i$ th case of Euler configuration is denoted by  $E_i$ . The motion of the three body constructs an  $i$ th Euler configuration  $E_i$  means the three bodies are on the same line and the mass with index is the middle one. In equal mass cases the mass between the others sits at the middle point of the line constructed by other two masses. The configuration the meridian on the shape surface which intersect with the configuration  $E_i$  is denoted by  $M_i$ . The solution between  $E_3$  and  $M_1$  called minimizing arc in the paper is provided by the variational method, and it is the start point of the orbit construction.

In the sixth part of the work of Alain Chenciner and Richard Montgomery [12], the Lemma 6 of their paper is the final result given by group theory. The Lemma served as the proof of the third property in the theorem and provided the proof of the shape of the orbit with a solid base. The statement of the Lemma is as follows:

Lemma: After being symmetrized, a path  $x$  consists 12 pieces of minimizing arcs gives a loop of  $T$  periodic which is of the form  $x(a) = (p(a + \frac{2T}{3}), p(a + \frac{T}{3}), p(a))$ .

Since the page limitation, the lemma would not be fully proved in this paper. However, the author is thrilled to show the second step of the proof of the lemma, in which the group action was used to construct the full orbit from 12 pieces.

To prove the Lemma, Chenciner and Montgomery had first given a solution from  $E_3$  to  $M_1$  which is called minimizing arc. Then, they proved the orthogonality of the minimum arc. The orthogonality guaranteed that the orbits can fit together smoothly after reflection. Then, the 12 subarcs congruent to the initial minimizing solution arc can construct the whole orbit.

Due to the properties of the minimizing arcs, a reflection  $s_1$  can be constructed to continue the solution, thus construct a solution from  $E_3$  and travel through  $E_1$ ,  $E_2$  and finally back to itself. The reflection  $s_1$  satisfies  $x(\frac{T}{12} + a) = s_1(x(\frac{T}{12} - a))$ .

Geometrically, one can consider the set of oriented triangles with the inertia 1, in the reduced space, such set construct the shape sphere. Then,  $M_i$  is meridian of the sphere. The reflection  $s_1$  can be seen as a reflection of  $M_1$  on the sphere.  $s_1$  can also be realized in the inertial plane with

the action of Klein group defined in last part of this paper. At the time  $\bar{T} = \frac{T}{12}$  the triangle constructed by the three bodies is a isosceles triangle of  $M_1$  type. By choosing to coordinate in  $\mathbb{R}^2$  so that  $\tau$ , the reflection symmetry of the triangle satisfies  $\tau \bullet (a, b) = (a, -b)$ . An operator  $S_1$  can be defined interchanging the masses 2 and 3 i.e.  $S_1(a_1, a_2, a_3) = (a_1, a_3, a_2)$ . Then  $s_1 = \tau \circ S_1 = S_1 \circ \tau$ . In this way, the solution from  $E_3$  to  $E_1$ . In order to continue the solution through  $E_2$  a new operator  $H$ , which represents a half twist of the arc, satisfying  $x(\frac{T}{6} - a) = H(x(\frac{T}{6} + a))$ . Geometrically, this is the composition of the reflection about the meridian  $M_2$  and about the equator of the sphere. To realize the operation in the inertial plane one can compose  $\sigma \circ \tau \bullet (a, b) = (-a, -b)$  and the interchange of the mass 1 and the mass 3.

Continuing the solution in above way with appropriate half twist and reflection, a smooth curve consists of 12 congruent arcs is constructed. Such curve is a solution curve  $\frac{T}{12}$ -periodic mod rotation.

Finally in the third step of the proof, the authors reconstruct the full solution from the solution curve on the reduced space by using the area rule [12]. In the process the symmetry also plays an important role.

In the proof, the action of the Klein group is of primary importance in the proof idea. The equal mass planar three-body problem has a lot of symmetries, which make simple and elegant proof possible. The essence of the method is the observation of the geometrical symmetry on the shape sphere and the orthogonality of the minimizing arc. This observation shows that group theory is an excellent way to describe the symmetrical phenomenon thus reducing many dynamical problems to easier ones.

## 4. Conclusion

In this paper, the author introduced the application of group action in solving complex differential equations and specific physical problems. The proof of the Lemma demonstrates the great power of group theory on reducing the computation and describing symmetries. In the problems concerning a lot of variables and with good geometrical properties, like the equal mass three-body problem discussed in this paper, the appropriate choice of group action and symmetry groups can reduce the phase space of the differential equations, thus visualize the problem and simplify the formulas. In the problems requiring periodic solutions, one can generate the whole solution from a piece of solution with the help of symmetry. In the future, group action can be used in more problems concerning periodic motions of dynamic systems.

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