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Combinatorial Invariants of Stable Curves in Genus 4: Classification and Computation

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Abstract: This paper focuses on the combinatorial invariants of stable curves by studying their associated pm-graphs on genus 4. These invariants, including $\varphi(\Gamma)$, $\varepsilon(\Gamma)$, and $\lambda(\Gamma)$, play a key role in the arithmetic intersection theory developed by Zhang and others. We begin by classifying all possible pm-graphs of genus 4 using a combinatorial approach and proving bounds on the number of vertices. Building on this classification, we introduce algorithmic methods to compute the admissible invariants of such graphs. Our approach relies on translating geometric data into graph-theoretic quantities such as edge lengths and vertex valences, and utilizes electrical network analogies like mostly resistance functions and Laplacian matrices to calculate $\tau(\Gamma)$ and $\theta(\Gamma)$, which are defined in the expressions of the admissible invariants. While our algorithms are applied concretely to genus 4, the methods developed here are adaptable to higher genera and broader families of graphs. This work not only contributes to the computation of local invariants in Zhang's decomposition formula but also lays groundwork for future automated classification and theoretical exploration in arithmetic geometry and its combinatorial counterparts.

Keywords: Stable Curves, Polarized Metrized Graphs, Admissible Invariants, Arithmetic Intersection Theory, Genus 4 Curve Classification

1. Introduction

Height theory is an important tool in number theory and arithmetic geometry. It provides a way to measure the "size" or "complexity" of algebraic objects, such as points on algebraic varieties or solutions to Diophantine equations. Our work arises from the application of height theory in arithmetic intersection theory, where it provides valuable information about algebraic cycles in geometric objects—natural objects to study, but difficult to analyze.

Let us first review the history of intersection theory. Based on the traditional intersection theory for algebraic surfaces, Arakelov developed an arithmetic intersection theory in [1], which addresses the non-compactness of Spec(Z). Faltings made significant contributions to this theory in [2] and ultimately used it to prove the Mordell conjecture. In 1992, Zhang established admissible intersection theory (see [3]). Later, in 2008, Zhang proved a remarkable theorem about the height of the Gross-Schoen cycles (see [4]).

Theorem 1.1. *Let X be a curve of genus $g>1$ over a field k, which is either a number field or the fraction field of a curve B. Then*

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$$\langle\Delta_e,\Delta_e\rangle=\frac{2g+1}{2g-2}\omega_a^2+6(g-1)\langle x_e,x_e\rangle-\sum_v\varphi(X_v)\log\mathrm{RN}(v), \tag{1}$$

where $\langle x_e,x_e\rangle$ *is the Néron-Tate height of the class* $e-K_X/(2g-2)$ *in* $\mathrm{Pic}^0(X)_{\mathbb{Q}}$*, and* φ_v *represents contributions from places* v *of* K*.*

In this paper, we focus on the $\varphi(\cdot)$ invariants arising from non-Archimedean places in the summation. More precisely, if v is a non-Archimedean place, then

$$\varphi(X_v)=-\frac{1}{4}\delta(X_v)+\frac{1}{4}\int_{\Gamma(X_v)}g_v(x,x)\left((10g+2)d\mu_a-\delta_{K_{X_v}}\right), \tag{2}$$

where $\delta(X_v)$ is the number of singular points (counted with multiplicity) on the special fiber of X_v, and $g_v(x,y)$ is the admissible Green function for the admissible metric $d\mu_v$. Here, K_{X_v} is the canonical divisor on the dual graph $\Gamma(X_v)$. In particular, $\varphi_v=0$ if X has good reduction at v.

This is a local-global theorem since the main quantity (the height of the Gross-Schoen cycle) is decomposed into a sum of contributions from local data. Expanding our understanding of this local information can help advance the study of the arithmetic behavior of algebraic curves.

Our paper primarily follows the approach of [5], which explicitly computes admissible invariants for genus 3 curves. In this work, we aim to compute the admissible invariants of curves over number fields, focusing on finite places.

2. Prerequisites

In this section, we give a short introduction to fundamental notions.

Definition 2.1. *A discrete valuation ring (DVR) A is a principal ideal domain that satisfies the condition that A has exactly one nonzero prime ideal.*

Example:

- ***p*-adic integers** $\mathbb{Z}_p$: For a prime p, the ring of p-adic integers $\mathbb{Z}_p$ is a classic example of a DVR.

The fraction field of a DVR is a discrete valuation field.

Let K be a field, and consider the vector space K^{n+1}. Take the set $K^{n+1}\setminus\{0\}$ and define an equivalence relation $\sim$ by:

$$x\sim y\Leftrightarrow\exists\lambda\in K^\times\ (\lambda\neq0): y=\lambda x. \tag{3}$$

The **projective *n*-space** $\mathbb{P}^n(K)$ is the set of these equivalence classes:

$$\mathbb{P}^n(K)=\left(K^{n+1}\setminus\{0\}\right)/\sim. \tag{4}$$

Each equivalence class corresponds to a **line** in K^{n+1}.

An ideal I in a graded ring R is said to be **homogeneous** if for every element $f\in I$, all the homogeneous components of f also lie in I. Precisely, if f can be written as a sum of its homogeneous parts:

$$f=f_{d_0}+f_{d_1}+f_{d_2}+\ldots+f_{d_r} \tag{5}$$

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where each f_{d_i} is a homogeneous polynomial of degree d_i, then I being homogeneous means that each f_{d_i} is itself in I. The quotient ring $\frac{R}{I}$ defines a projective variety, which is a geometric object.

Definition 2.2. *A **curve** is a projective variety of dimension 1.*

A curve, as defined here, can have several components. For example, the equation $XY=0$ defines a projective curve in the projective plane, which consists of two straight lines.

A point on the curve is **smooth** if its Jacobian has rank 1. A curve can also have singularities. For example, locally speaking, if the singular point resembles $xy=0$ near the origin, we call it a nodal singularity. This type of singular point will play an important role in this paper. A curve without singular points is called a smooth curve.

Fixing a DVR R, let $\mathrm{Frac}(R)$ be its fraction field, and let $k=R/\mathrm{m}$ be its residue field. Given a curve over $\mathrm{Frac}(R)$, we are concerned with its model (which has defining polynomials with coefficients in R). We are especially interested in **stable models**. This means that when considering the curve over R as a curve over k:

1. All singular points are nodal points;
2. All components of geometric genus 0 contain at least three nodal points;
3. All components of geometric genus 1 contain at least two nodal points.

Locally, these nodal singular points arise from equations like $xy\text{-}p^n=0$. We will call n the **thickness** at this point.

An important property of curves over a discrete valuation field is that, after some extension, they always have a stable model over the discrete valuation ring, which is also known as the stable reduction theorem.

Theorem 2.3. *Let R be a discrete valuation ring with fraction field K. Let C be a smooth projective curve over K with $H^0(C,\mathrm{O}_C)=K$ and genus $g\geq 2$. Then there exists an extension of discrete valuation rings $R\subset R'$ inducing a finite separable extension of fraction fields K'/K and a stable family of curves $Y \to \textup{Spec}(R')$ of genus g with $Y_{K'}\cong C_{K'}$ over K'.*

This theorem implies that stable models do exist, and this will serve as the foundation for the next section, where a curve is associated with a *pm*-graph constructed from its stable models.

3. Dual graph

When studying a stable curve X and its metric degeneration, we often consider the **dual graph** arising from the special fiber of the curve degeneration. In this section, we introduce the notions related to the dual graph and explain how they are involved in Zhang's decomposition formula at the end.

Let X be a projective algebraic curve over a DVR R with a smooth generic fiber. The stable reduction theorem states that there exists a discrete valuation ring extension R' of R such that X has a stable model X over $\mathrm{Frac}(R')$. As the curve degenerates, the special fiber of X may contain certain nodal singularities (intersection points). More precisely, these singular points can locally be expressed by $\frac{R'[x,y]_{\mathrm{m}}}{(xy\text{-}p^n)}$, where p is the uniformizer of R' and n is called the **thickness** of this nodal point, which does not depend on the choice of the stable model.

Now we can construct a graph from these data. Let the nodes correspond to **edges** denoted by E or $E(\Gamma)$, and let the irreducible components correspond to **vertices** denoted by V or $V(\Gamma)$. In addition, we define the length of each edge of Γ by the thickness of the nodal point. Then Γ can be regarded as a **metrized graph**. We further specify a polarization function over Γ by $\mathbf{q}:\Gamma\to\mathrm{R}$ based on the geometric genus of the components of the special fiber. The symbol $\mathrm{val}(\cdot)$ denotes the valence of a vertex (the number of semi-edges attached to it).

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The **canonical divisor** K of the pair $(\Gamma,\mathbf{q})$ is written as:

$$K=\sum_{p\in V(\Gamma)}\left(\mathrm{val}(p)-2+2\mathbf{q}(p)\right)p, \tag{6}$$

where $V(\Gamma)$ is the vertex set of the metrized graph Γ. If all the coefficients of the canonical divisor are positive, then we call the pair $(\Gamma,\mathbf{q})$ a pm-graph (**polarized metrized graph**). For simplicity, we will also denote a pm-graph by Γ from now on.

Once Γ is fixed, one can define various geometric and arithmetic invariants on this graph. Based on Zhang's work [4] on families of arithmetic curves, the **admissible invariants** of interest include $\epsilon(\Gamma)$, $\varphi(\Gamma)$, and $\lambda(\Gamma)$.

In [3], Zhang constructed a Green function g_{ad} and an admissible measure μ_{ad} on Γ, and the admissible invariants are defined as follows:

$$\begin{aligned}
\varphi(\overline{\Gamma}) &= -\frac{1}{4}\delta(\Gamma)+\frac{1}{4}\int_\Gamma g_\mu(x,x)\big((10g+2)\mu(x)-\delta_K(x)\big),\\
\epsilon(\overline{\Gamma}) &= \int_\Gamma g_\mu(x,x)\big((2g-2)\mu(x)+\delta_K(x)\big),\\
\lambda(\overline{\Gamma}) &= \frac{g-1}{6(2g+1)}\varphi(\overline{\Gamma})+\frac{1}{12}\left(\delta(\Gamma)+\epsilon(\overline{\Gamma})\right),
\end{aligned} \tag{7}$$

where δ denotes the total length of the graph and δ_K denotes the Dirac measure associated with K.

This method is abstract and difficult to handle directly. Later, Baker–Rumely studied the Laplacian theory on metrized graphs and provided a more explicit way of computing these admissible invariants (see [6]). They introduced two new invariants with physical meaning. More precisely, we take the length of an edge as its electrical resistance, and define:

$$\begin{aligned}
\tau(\Gamma) &= \frac{1}{4}\int_\Gamma \left(\frac{\partial r(x,y)}{\partial x}\right)^2 dx,\\
\theta(\Gamma) &= \sum_{p,q\in V(\Gamma)}\left(\mathrm{val}(p)-2+2\mathbf{q}(p)\right)\left(\mathrm{val}(q)-2+2\mathbf{q}(q)\right)r(p,q),
\end{aligned} \tag{8}$$

where $r(x,y)$ denotes the equivalent resistance function between x and y on Γ.

Let g be the first Betti number of the graph ($g(\Gamma)=E-V+1$). We define the **genus** $\overline{g}(\Gamma)$ of a pm-graph Γ as:

$$\overline{g}(\Gamma)=g(\Gamma)+\sum_{p\in V(\Gamma)}\mathbf{q}(p). \tag{9}$$

Now, let Γ be a pm-graph of genus 4. Then we have:

$$\begin{aligned}
\epsilon(\Gamma) &= 3\tau(\Gamma)+\frac{\theta(\Gamma)}{8},\\
\varphi(\Gamma) &= \frac{9\tau(\Gamma)}{2}+\frac{\theta(\Gamma)}{16}-\frac{\delta(\Gamma)}{4},\\
\lambda(\Gamma) &= \frac{\tau(\Gamma)}{2}+\frac{\theta(\Gamma)}{72}+\frac{5\delta(\Gamma)}{72},
\end{aligned} \tag{10}$$

according to [7].

Let X be a stable model of X at a finite place v. Then the dual graph of this model is a pm-graph. This is why we have established the above framework. In the following sections, we will try to classify the pm-graphs in genus 4 and develop an algorithmic method for computing these invariants.

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4. Classification

In Section 3, we define the dual graph and its genus. The study of a dual graph can provide insights into the $\varphi(\cdot)$ invariants appearing in Zhang's decomposition formula (Equation 1). There has been previous work on the $\langle\Delta_e,\Delta_e\rangle$ for curves in genus ≤ 3 (see [8] and [5]). In this section, we classify all admissible invariants for genus 4 curves in the spirit of Cinkir's work [5].

The φ invariant in Zhang's equation is derived from the stable model of X at finite places s. According to Section 3, we need to list all the principally polarized graphs such that

$$\overline{g}(\Gamma)=g(\Gamma)+K=4, \tag{11}$$

$$\mathbf{q}(v)+2\cdot(\mathrm{val}(v)-2)>0 \tag{12}$$

where g is the first Betti number and K is the degree of the canonical divisor.

Although an enumerative approach is possible, the following proposition greatly simplifies the work.

Proposition 4.1. *A pm-graph in genus 4 has at most 6 vertices.*

Proof. If the graph has a vertex with non-zero genus, we can add a self-loop to it and reduce the polarization at this vertex by 1. Repeating this procedure, we can obtain a pm-graph without reducing the number of vertices. Thus, we can assume that the graph only has vertices with zero polarization.

Each such vertex will have valence at least 3. Therefore, the graph has at least $3V/2$ edges, where V is the cardinality of the vertex set. We then have:

$$4=g(\Gamma)+K=g(\Gamma)=E-V+1\geq 3V/2-V+1=V/2+1. \tag{13}$$

This implies $V\leq 6$.

Thus, we only need to consider graphs with no more than 6 vertices. The following Table 1 and Table 2 list all the possibilities. The thickness (lengths of edges) are omitted in the graphs. In the next section, we will present an algorithmic method for computing the admissible invariants associated with these dual graphs, under the condition that these lengths are known. These graphs are listed in the remaining pages of this section.

Table 1. Table of $\overline{\Gamma}$, $\delta(\overline{\Gamma})$, $\theta(\overline{\Gamma})$ and $\tau(\overline{\Gamma})$

serial number	Γ	serial number	Γ
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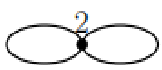
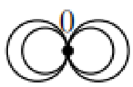
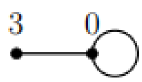
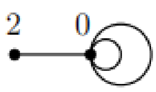
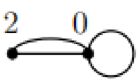
1III		1IV	
1V		2I	
2II		2III	
2IV		2V	

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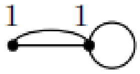
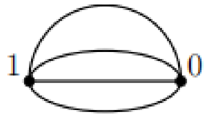
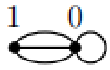
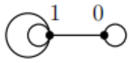
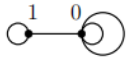
Table 1: (continued).

serial number	Γ	serial number	Γ
2VI		2VII	
2VIII		2IX	
2X		2XI	
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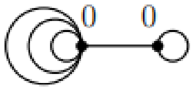
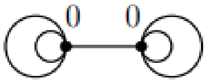
2XIV		2XV	
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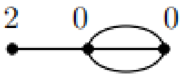
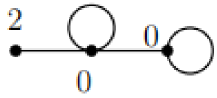
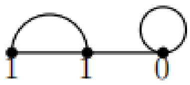
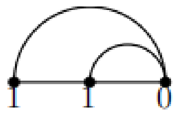
3I		3II	
3III		3IV	

Table 2. Table 1 continued

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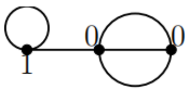
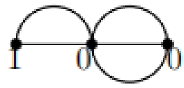
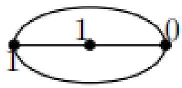
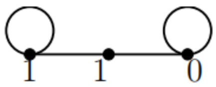
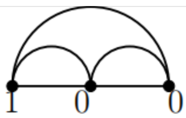
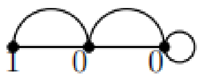
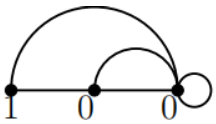
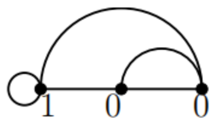
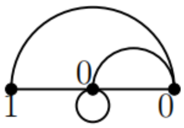
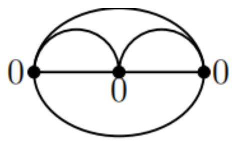
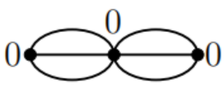
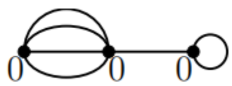
3V		3VI	
3VII		3VIII	
3IX		3X	
3XI		3XII	
3XIII		3XIV	
3XV		3XVI	

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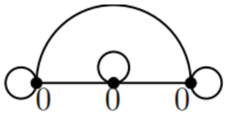
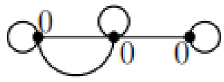
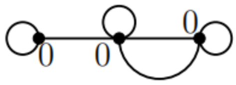
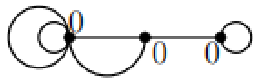
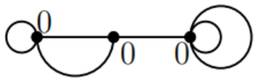
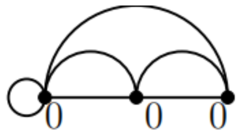
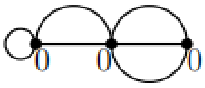
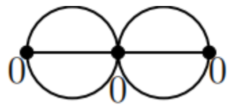
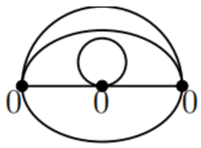
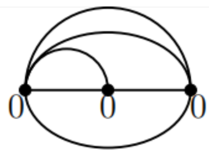
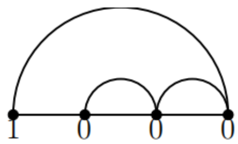
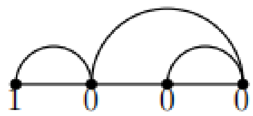
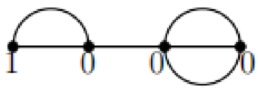
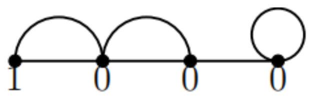
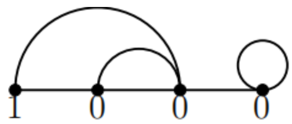
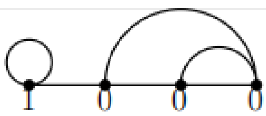
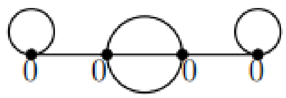
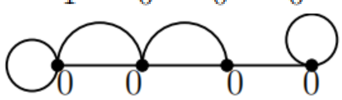
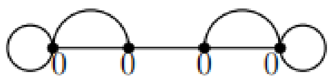
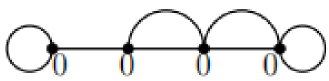
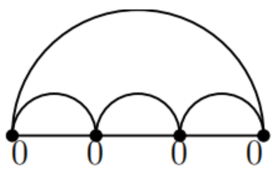
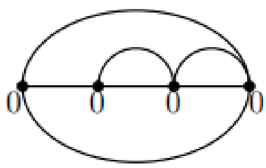
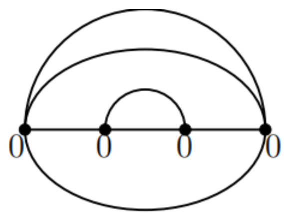
Table 2. (continued).

3XVII		3XVIII	
3XIX		3XX	
3XXI		3XXII	
3XXIII		3XXIV	
3XXV		3XXVI	
4I		4II	
4III		4IV	
4V		4VI	
4VII		4VIII	
4IX		4X	
4XI		4XII	
4XIII		4XIV	

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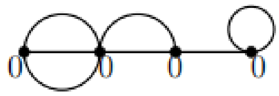
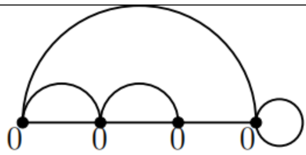
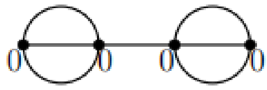
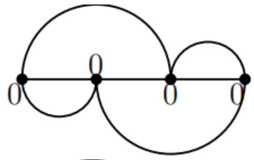
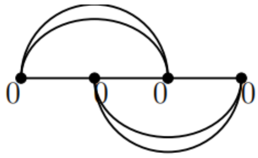
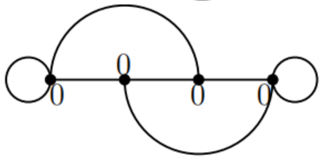
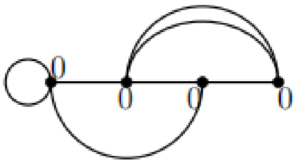
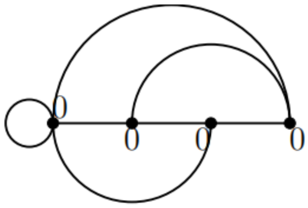
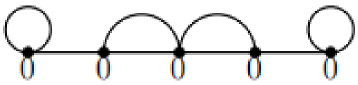
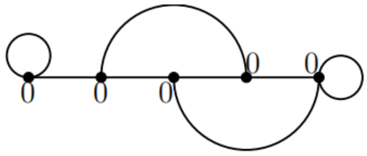
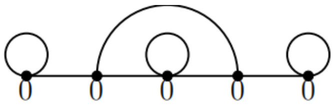
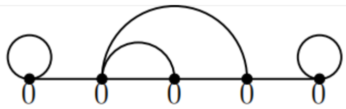
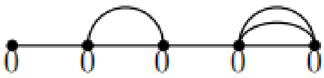
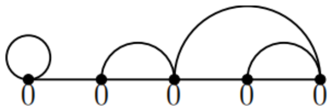
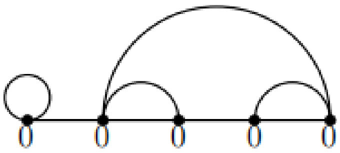
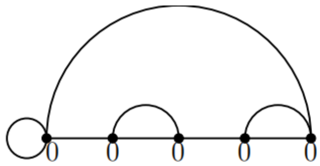
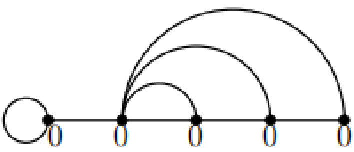
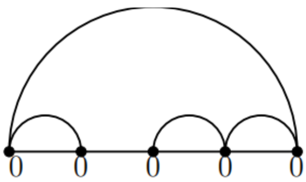
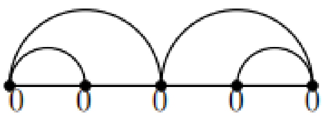
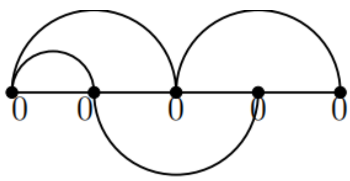
4XV		4XVI	
4XVII		4XVIII	
4XIX		4XX	
4XXI		4XXII	
5I		5II	
5III		5IV	
5V		5VI	
5VII		5VIII	
5IX		5X	
5XI		5XII	

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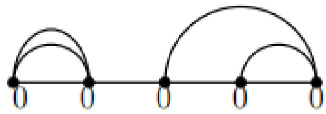
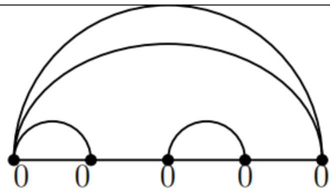
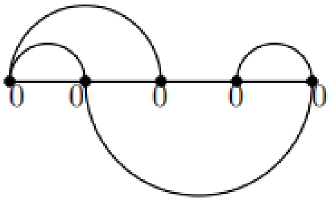
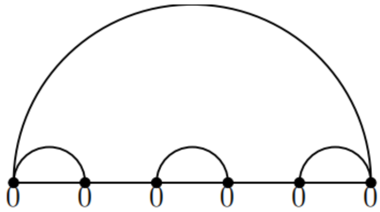
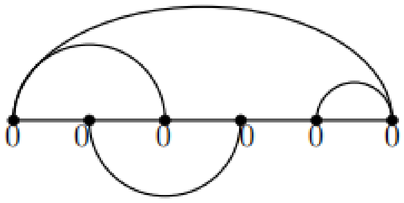
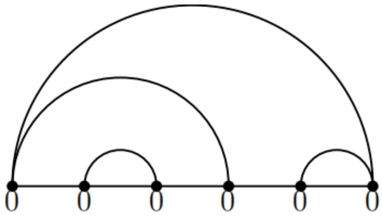
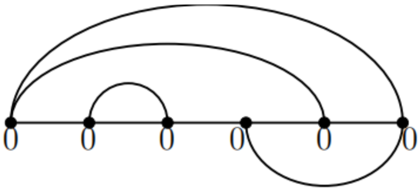
5XIII		5XV	
5XV		6I	
6II		6III	
6IV			

5. Computation

In the last section, we reduced the computation of $\epsilon(\Gamma)$, $\varphi(\Gamma)$, and $\lambda(\Gamma)$ to that of $\delta(\Gamma)$, $\theta(\Gamma)$, and $\tau(\Gamma)$. In this section, we will present an algorithmic approach to compute the admissible invariants defined in the previous section.

Algorithm 1 Computation of $\delta(\Gamma)$

Input: V: the set of vertices of Γ
E: the metrized edge set of Γ

Output: $\delta(\Gamma)$
1: $\delta(\Gamma)$=0
2: **for** each edge e in E **do**
3: $\delta(\Gamma)$+=length(e)
4: **end for**
5: **return** $\delta(\Gamma)$

The invariant $\delta(\Gamma)$ is the easiest one to compute, as we only need to sum the lengths of all the edges. Next, we compute $\theta(\Gamma)$. In the following pseudo-algorithm, we use a function $r(x,y)$ that computes the electric resistance between x and y in the circuit. The explicit code for $r(x,y)$ is displayed later.

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Algorithm 2 Computation of $\theta(\Gamma)$

Input: V: the set of vertices of Γ
E: the metrized edge set of Γ
val(): the valence function on the vertices
q(): the polarization function on the vertices
$r(x,y)$: the electric resistance function on Γ

Output: $\theta(\Gamma)$
1: $\theta(\Gamma)$=0
2: **for** each v_1 in V **do**
3: **for** each v_2 in $V\setminus\{v_1\}$ **do**
4: $\theta(\Gamma)$+=$(\mathrm{val}(v_1)\text{-}2+2\mathrm{q}(v_1))\cdot(\mathrm{val}(v_2)\text{-}2+2\mathrm{q}(v_2))\cdot r(v_1,v_2)$
5: **end for**
6: **end for**
7: **return** $\theta(\Gamma)$

The algorithm above relies on the resistance function on the graph. The code for this function can be found in the Appendix. The algorithm is based on a well-known method which computes the equivalent resistance by applying the Laplacian matrix.

Finally, we need to compute $\tau(\Gamma)$. Although its expression involves partial differentiation and integration over the metrized graph, we do not need to construct a new algorithm from scratch. Considering that $\varphi(\Gamma)$ does not depend on the initialization of the second variable y, we can simply set it to be a vertex Y_0 of the dual graph. As x moves along a certain edge e of the graph, we can divide e into two subedges e_1 and e_2, where:

$$\mathrm{length}(e_1)=x, \quad \mathrm{length}(e_2)=\mathrm{length}(e)\text{-}x.$$

Thus, the electrical resistance function on this edge can be expressed by $r(x,Y_0)=r'(x,Y_0)$, where r' is the resistance function on the new metrized graph Γ' obtained by inserting a new vertex at the junction of e_1 and e_2. Since our resistance algorithm accepts symbolic inputs, we can compute its partial derivative and then integrate it symbolically. Iterating this operation over all the edges will lead to the formula for $\tau(\Gamma)$.

Algorithm 3 Computation of $\tau(\Gamma)$

Input: V: the set of vertices of Γ
E: the metrized edge set of Γ
val(): the valence function on the vertices
q(): the polarization function on the vertices
$r(x,y)$: the electric resistance function on Γ

Output: $\tau(\Gamma)$
1: $\tau(\Gamma)$=0
2: Fix a vertex point Y_0
3: **for** each edge e in E **do**
4: Compute the resistance function $r'(x,Y_0)$ on Γ'
5: Take the formal derivative of $r'(x,Y_0)$ with respect to x

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6: Compute $\phi_e(\Gamma)$ defined by $\int_e r'_x(x,Y_0)^2$
7: $\tau(\Gamma)$+=$\phi_e(\Gamma)$
8: **end for**
9: **return** $\tau(\Gamma)$

6. Summary and exploration

In this paper, we have studied the dual graphs for curves of genus 4. First, we classified all possible types of dual graphs for such curves. We then provided an algorithm for the explicit computation of admissible invariants. Notably, the algorithm is not restricted to genus 4 and can be extended to other genera, making it a valuable tool for the study of admissible invariants in a broader context.

Based on the findings in this paper, there are several promising directions for future research:

- **Automating the classification for arbitrary genera:** The classification we provided is essentially an enumerative approach. It would be interesting to develop an algorithm that can automatically generate all possible dual graph types for any given genus g. Such a tool would streamline the analysis of stable curves across different genera and further advance our understanding of the combinatorial structure of dual graphs.
- **Lower bounds for electrical resistance:** It is known that the admissible invariant $\phi(\Gamma)$ is positive when Γ is non-trivial, and it is closely related to the electrical resistance of the graph. A natural question is whether these invariants can be used to derive lower bounds for electrical resistance in a graph-theoretic context. This could offer an interesting intersection between pure mathematics and physics, providing fresh insights into both fields.
- **Concrete curve construction:** While we have classified the possible dual graphs for stable genus 4 curves, it remains an open question whether every such graph can be realized by an explicit curve. A systematic method for constructing concrete curves corresponding to a given dual graph would provide a deeper understanding of the relationship between combinatorial invariants and geometric objects. This would also allow for more concrete applications of the dual graph theory in algebraic geometry.

Overall, the results of this paper lay a foundation for further exploration into the combinatorial and arithmetic aspects of stable curves, with connections to both mathematical and physical theories. We hope these directions will inspire future work in the area and contribute to the growing body of knowledge on the topic.

7. Code for the resistance function

```
import numpy as np

class ElectricalNetwork:
    def __init__(self, N, resistors):
        """
        Initialize an electrical network with N nodes.
        param N: Number of nodes
        param resistors:
        List of resistors, where each resistor is represented as
(a, b, R) meaning a resistor of value R is connected between nodes
a and b.
        """
        self.N = N
```

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```
        self.adj_matrix = {i: {} for i in range(N)}
        self.add_resistors(resistors)

    def add_resistors(self, resistors):
        """
        Add a list of resistors to the network.
        If multiple resistors exist between the same two nodes,
compute the parallel equivalent resistance.
        """
        for a, b, r in resistors:
            if b in self.adj_matrix[a]:
                # If a resistor already exists between (a, b),
compute the parallel equivalent resistance
                self.adj_matrix[a][b] = 1 / (1 /
self.adj_matrix[a][b] + 1 / r)
                self.adj_matrix[b][a] = self.adj_matrix[a][b]
            else:
                self.adj_matrix[a][b] = r
                self.adj_matrix[b][a] = r

    def compute_effective_resistance(self, node1, node2):
        """
        Compute the effective resistance between node1 and node2
using the Laplacian pseudo-inverse.
        """
        if node1 == node2:
            return 0

        # Construct the Laplacian matrix
        L = np.zeros((self.N, self.N))
        for i in range(self.N):
            for j in self.adj_matrix[i]:
                L[i, j] -= 1 / self.adj_matrix[i][j]
                L[i, i] += 1 / self.adj_matrix[i][j]

        # Compute the pseudo-inverse of the Laplacian matrix
        L_pinv = np.linalg.pinv(L)

        # Effective resistance formula: R_eq = MPL[i,i] + MPL[j,j]
- 2 * MPL[i,j]
        return L_pinv[node1, node1] + L_pinv[node2, node2] - 2 *
L_pinv[node1, node2]
```

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