

Existence and Regularity of Solutions to Degenerate Nonlinear p -Laplacian Elliptic Equations with a Low-Order term

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Abstract: In this paper we study the existence and regularity of solutions for a class of nonlinear degenerate elliptic equations

$$\begin{cases} -\operatorname{div} \left(\frac{a(x, \nabla u)}{(1+u)^{\gamma(p-1)}} \right) + u = f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Here $1 < p < 2$ and $\gamma \geq 0$, while $f(x) \geq 0$ is a function in some Lebesgue space. Due to the presence of nonlinear terms, traditional variational methods are not applicable. Therefore, this paper begins by utilizing a truncation technique to derive an approximate equation, from which the existence and non-negativity of the approximate solution are obtained using Lions' results and estimates. The approximate solution is then estimated using test functions, Hölder inequality, Fatou lemma, and other methods. This estimation ensures that the solution to the original problem can be obtained through the passage to the limit from the solution of the approximating problem. Finally, the existence of the solution to the original problem is proven using the control convergence theorem, Vitali's theorem, and other related methods, leading to the uniqueness of the solution.

Keywords: Regularizing effects; Degenerate elliptic equations; Convection term

1. Introduction and statement of the results

In recent years, some existence and regularity results of the solutions to the Dirichlet problem

$$\begin{cases} -\operatorname{div} \left(\frac{\nabla u}{(1+|u|)^\theta} \right) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1)$$

depending on the summability of the term f have been proved in [1,2,3,4], for example in [3], when $0 < \theta \leq 1$ and $f \in L^m(\Omega)$ with $m \geq 1$, existence results show that solutions lie in the Sobolev space $W_0^{1,q}(\Omega)$ with $q \leq 2$ if f is regular enough, or are entropy solutions. For $\theta > 1$, a non-existence result for constant sources has been established in [5](where the principal part is nonlinear with respect to

the gradient), even for $L^\infty(\Omega)$ data f . Therefore, if the operator becomes “too degenerate”, existence may be lost even for data expected to give bounded solutions.

Furthermore, [6](see also in [4,7,8]) has demonstrated that the inclusion of a lower-order term enhances the existence and regularity of the solutions, the elliptic equation as follows:

$$\begin{cases} -\operatorname{div} \left(\frac{a(x, \nabla u)}{(1+|u|)^\gamma} \right) + u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (2)$$

where $\gamma > 0$, f belongs to some Lebesgue spaces, and $a : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function, then there exists distributional solution u in $W_0^{1,1}(\Omega) \cap L^{\frac{\gamma+2}{2}}(\Omega)$, while f belongs to $L^{\frac{\gamma+2}{2}}(\Omega)$, this result improves upon the entropy solution and its integrability obtained in (1).

In this paper, we are going to deal with existence of positive solutions for the following elliptic equation

$$\begin{cases} -\operatorname{div} \left(\frac{a(x, \nabla u)}{(1+u)^{\gamma(p-1)}} \right) + u = f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \quad (3)$$

The main contribution of this work is to extend the results from [6] to the p-Laplace operator while $1 < p < 2$. Additionally, due to the non-negativity of f , we establish the non-negativity of the solution u .

We now state the assumptions on the various terms in (3):

- Ω is a bounded, open subset of $\mathbb{R}^N$, $N > 2$;
- $a : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function (i.e., $a(x, \cdot)$ is continuous on $\mathbb{R}^N$ for almost every x in Ω , and $a(\cdot, \xi)$ is measurable on Ω for every ξ in $\mathbb{R}^N$) such that

$$a(x, \xi) \cdot \xi \geq \alpha |\xi|^p, \quad (4)$$

$$|a(x, \xi)| \leq \beta |\xi|^{p-1}, \quad (5)$$

$$[a(x, \xi) - a(x, \eta)] \cdot (\xi - \eta) > 0, \quad (6)$$

for almost every x in Ω and for every ξ and η in $\mathbb{R}^N$, $\xi \neq \eta$, where α and β are positive constants;

- p and γ are real numbers such that

$$1 < p < 2, \quad (7)$$

$$\gamma \geq 0; \quad (8)$$

- $f(x) \geq 0$ is a function in some Lebesgue space $L^m(\Omega)$.

Our main result is the following.

Theorem 1. *Let $\gamma \geq 0$, and $f \geq 0$ is a function in $L^{\frac{\gamma(p-1)+2}{p}}(\Omega)$. Then there exists a distributional solution u in $W_0^{1,1}(\Omega) \cap L^{\frac{\gamma(p-1)+2}{p}}(\Omega)$ of (3), that is*

$$\int_{\Omega} \frac{a(x, \nabla u) \cdot \nabla \psi}{(1+u)^{\gamma(p-1)}} + \int_{\Omega} u \psi = \int_{\Omega} f \psi, \quad \forall \psi \in W_0^{1,\infty}(\Omega). \quad (9)$$

The paper is organized as follows. In Section 2. we introduce the existence and non-negativity of solutions of the approximating problems; in Section 3. we discuss a priori estimates and provide convergence analyses necessary to prove the main result. At last, Section 4. is devoted to address the uniqueness of the solution and the non-existence results under certain special conditions.

2. Approximating solutions

We will make use of the following function of one real variable, defined for s in $\mathbb{R}$ and $k > 0$:

$$T_k(s) = \max(-k, \min(s, k))$$

Our first result deals with the existence of solutions for an approximating problem.

Lemma 1. *Let $g(x) \geq 0$ be a function in $L^\infty(\Omega)$, then there exists $v \geq 0$ solution of Dirichlet problem*

$$\begin{cases} -\operatorname{div} \left(\frac{a(x, \nabla v)}{(1+|v|)^{\gamma(p-1)}} \right) + v = g, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases} \quad (10)$$

Proof. Let $M = \|g\|_{L^\infty(\Omega)} + 1$, and consider the problem

$$\begin{cases} -\operatorname{div} \left(\frac{a(x, \nabla v)}{(1+|T_M(v)|)^{\gamma(p-1)}} \right) + v = g, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases} \quad (11)$$

Since the differential operator is pseudomonotone and coercive thanks to the assumptions on a and to the truncature, by the results of [9] there exists a weak solution v in $W_0^{1,p}(\Omega)$ of (11). Choosing $(|v| - \|g\|_{L^\infty(\Omega)})_+ \operatorname{sgn}(v)$ as test function we obtain, dropping the nonnegative first term,

$$\int_{\Omega} |v| (|v| - \|g\|_{L^\infty(\Omega)})_+ \leq \int_{\Omega} \|g\|_{L^\infty(\Omega)} (|v| - \|g\|_{L^\infty(\Omega)})_+.$$

Thus,

$$\int_{\Omega} (|v| - \|g\|_{L^\infty(\Omega)}) (|v| - \|g\|_{L^\infty(\Omega)})_+ \leq 0,$$

so that $|v| \leq \|g\|_{L^\infty(\Omega)} < M$. Therefore, $T_M(v) = v$, and v is a bounded weak solution of (10). Let now $f_n \geq 0$ be a sequence of $L^\infty(\Omega)$ functions which converges to f in $L^{\frac{\gamma(p-1)+2}{p}}(\Omega)$, and such that $f_n \leq f$ almost everywhere in Ω , and consider the approximating problem

$$\begin{cases} -\operatorname{div} \left(\frac{a(x, \nabla u_n)}{(1+|u_n|)^{\gamma(p-1)}} \right) + u_n = f_n(x), & \text{in } \Omega, \\ u_n = 0, & \text{on } \partial\Omega. \end{cases} \quad (12)$$

A solution u_n in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ exists choosing $g = f_n$ in (10).

Next we will prove $u_n \geq 0$, following the techniques of [10], choosing $T_k(u_n)$ as test function of (12), using (4) we get

$$\alpha \int_{\{|u_n| \leq k\}} \frac{|\nabla u_n|^p}{(1+|u_n|)^{\gamma(p-1)}} + \int_{\Omega} u_n T_k(u_n) \leq \int_{\Omega} f_n T_k(u_n). \quad (13)$$

Dividing by k , and dropping a positive term, we obtain

$$\int_{\Omega} \frac{u_n T_k(u_n)}{k} \leq \int_{\Omega} \frac{|f_n| |T_k(u_n)|}{k}.$$

Letting h tend to zero, we thus get

$$\int_{\Omega} |u_n| \leq \int_{\Omega} |f_n|. \quad (14)$$

With the same calculations, choosing $T_k(u_{n-})$ as test function, and using that $f_n \geq 0$, one obtains

$$0 \leq \int_{\Omega} u_{n-} \leq 0,$$

which implies that $u_n \geq 0$, so that (12) becomes

$$\begin{cases} -\operatorname{div} \left(\frac{a(x, \nabla u_n)}{(1+u_n)^{\gamma(p-1)}} \right) + u_n = f_n(x), & \text{in } \Omega, \\ u_n = 0, & \text{on } \partial\Omega. \end{cases} \quad (15)$$

3. A priori estimates and the proof of main result

Lemma 2. *If u_n is a solution to problem (15), then, for every $k \geq 0$,*

$$\int_{\{u_n \geq k\}} u_n^{\frac{\gamma(p-1)+2}{p}} \leq \int_{\{u_n \geq k\}} f^{\frac{\gamma(p-1)+2}{p}}; \quad (16)$$

$$\int_{\{u_n \geq k\}} \frac{|\nabla u_n|^p}{(1+u_n)^{\frac{\gamma(p-1)+2}{p}}} \leq C \left[\int_{\{u_n \geq k\}} f^{\frac{\gamma(p-1)+2}{p}} \right]^{\frac{p}{\gamma(p-1)+2}}; \quad (17)$$

$$\int_{\{u_n \geq k\}} |\nabla u_n| \leq C \left[\int_{\{u_n \geq k\}} f^{\frac{\gamma(p-1)+2}{p}} \right]^{\frac{1}{\gamma(p-1)+2}}; \quad (18)$$

$$\alpha \int_{\Omega} |\nabla T_k(u_n)|^p \leq k(1+k)^{\gamma(p-1)} \int_{\Omega} f. \quad (19)$$

Here, and in the following, C denotes a positive constant depending on $\alpha, \gamma, \operatorname{meas}(\Omega)$, and the norm of f in $L^{\frac{\gamma(p-1)+2}{p}}(\Omega)$.

Proof. Let $k \geq 0, h \geq 0$, and let $\psi_{h,k}(s)$ be the function defined by

$$\psi_{h,k}(s) = \begin{cases} 0 & \text{if } 0 \leq s \leq k, \\ h(s-k) & \text{if } k < s \leq k + \frac{1}{h}, \\ 1 & \text{if } s > k + \frac{1}{h}, \\ \psi_{h,k}(s) = -\psi_{h,k}(-s) & \text{if } s < 0. \end{cases}$$

Note that

$$\lim_{h \rightarrow +\infty} \psi_{h,k}(s) = \begin{cases} 1 & \text{if } s > k, \\ 0 & \text{if } |s| \leq k, \\ -1 & \text{if } s < -k. \end{cases}$$

Let $\varepsilon > 0$, and choose $(\varepsilon + u_n)^{\frac{\gamma(p-1)+2-p}{p}} \psi_{h,k}(u_n)$ as a test function in (15) (such a test function is admissible since u_n belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $\psi_{h,k}(0) = 0$). We obtain

$$\begin{aligned} & \frac{\gamma(p-1)+2-p}{p} \int_{\Omega} a(x, \nabla u_n) \cdot \nabla u_n \frac{(\varepsilon + u_n)^{\frac{\gamma(p-1)+2-p}{p}-1}}{(1+u_n)^{\gamma(p-1)}} \psi_{h,k}(u_n) \\ & + \int_{\Omega} \frac{a(x, \nabla u_n) \cdot \nabla u_n}{(1+u_n)^{\gamma(p-1)}} \psi'_{h,k}(u_n) (\varepsilon + u_n)^{\frac{\gamma(p-1)+2-p}{p}} \\ & + \int_{\Omega} u_n (\varepsilon + u_n)^{\frac{\gamma(p-1)+2-p}{p}} \psi_{h,k}(u_n) \\ & = \int_{\Omega} f_n (\varepsilon + u_n)^{\frac{\gamma(p-1)+2-p}{p}} \psi_{h,k}(u_n). \end{aligned} \quad (20)$$

By (4), and since $\psi'_{h,k}(s) \geq 0$, the first two terms are nonnegative, so that we obtain, recalling that $f_n \leq f$,

$$\int_{\Omega} u_n (\varepsilon + u_n)^{\frac{\gamma(p-1)+2-p}{p}} \psi_{h,k}(u_n) \leq \int_{\Omega} f (\varepsilon + u_n)^{\frac{\gamma(p-1)+2-p}{p}} \psi_{h,k}(u_n).$$

Letting ε tend to zero and h tend to infinity, we obtain, by Fatou's lemma (on the left hand side) and by Lebesgue's theorem (on the right hand side, recall that u_n belongs to $L^\infty(\Omega)$),

$$\int_{\{u_n \geq k\}} u_n^{\frac{\gamma(p-1)+2}{p}} \leq \int_{\{u_n \geq k\}} f u_n^{\frac{\gamma(p-1)+2-p}{p}}.$$

Using Hölder inequality on the right hand side we obtain

$$\int_{\{u_n \geq k\}} u_n^{\frac{\gamma(p-1)+2}{p}} \leq \left[\int_{\{u_n \geq k\}} f^{\frac{\gamma(p-1)+2}{p}} \right]^{\frac{p}{\gamma(p-1)+2}} \left[\int_{\{u_n \geq k\}} u_n^{\frac{\gamma(p-1)+2}{p}} \right]^{\frac{\gamma(p-1)+2-p}{\gamma(p-1)+2}}.$$

Simplifying equal terms we thus have

$$\int_{\{u_n \geq k\}} u_n^{\frac{\gamma(p-1)+2}{p}} \leq \int_{\{u_n \geq k\}} f^{\frac{\gamma(p-1)+2}{p}},$$

which is (16). Note that if $k = 0$, then (16) becomes

$$\int_{\Omega} u_n^{\frac{\gamma(p-1)+2}{p}} \leq \int_{\Omega} f^{\frac{\gamma(p-1)+2}{p}} = \|f\|_{L^{\frac{\gamma(p-1)+2}{p}}(\Omega)}^{\frac{\gamma(p-1)+2}{p}}. \quad (21)$$

Now we consider (20) written for $\varepsilon = 1$. Dropping the nonnegative second and third terms, and using that $f_n \leq f$, we have

$$\frac{\gamma(p-1)+2-p}{p} \int_{\Omega} \frac{a(x, \nabla u_n) \cdot \nabla u_n}{(1+u_n)^{\gamma(p-1)-\frac{\gamma(p-1)+2-p}{p}+1}} \psi_{h,k}(u_n) \leq \int_{\Omega} f (1+u_n)^{\frac{\gamma(p-1)+2-p}{p}} \psi_{h,k}(u_n).$$

Using (4), and letting h tend to infinity, we get (using again Fatou's lemma and Lebesgue's theorem)

$$\alpha \frac{\gamma(p-1)+2-p}{p} \int_{\{u_n \geq k\}} \frac{|\nabla u_n|^p}{(1+u_n)^{\frac{\gamma(p-1)^2+2p-2}{p}}} \leq \int_{\{u_n \geq k\}} f(1+u_n)^{\frac{\gamma(p-1)+2-p}{p}}.$$

Hölder inequality on the right hand then gives

$$\begin{aligned} \alpha \frac{\gamma(p-1)+2-p}{p} \int_{\{u_n \geq k\}} \frac{|\nabla u_n|^p}{(1+u_n)^{\frac{\gamma(p-1)^2+2p-2}{p}}} &\leq \left[\int_{\{u_n \geq k\}} f^{\frac{\gamma(p-1)+2}{p}} \right]^{\frac{p}{\gamma(p-1)+2}} \left[\int_{\{u_n \geq k\}} (1+u_n)^{\frac{\gamma(p-1)+2}{p}} \right]^{\frac{\gamma(p-1)+2-p}{\gamma(p-1)+2}} \\ &\leq \left[\int_{\{u_n \geq k\}} f^{\frac{\gamma(p-1)+2}{p}} \right]^{\frac{p}{\gamma(p-1)+2}} \left[\int_{\Omega} (1+u_n)^{\frac{\gamma(p-1)+2}{p}} \right]^{\frac{\gamma(p-1)+2-p}{\gamma(p-1)+2}}, \end{aligned}$$

so that, by (21),

$$\int_{\{u_n \geq k\}} \frac{|\nabla u_n|^p}{(1+u_n)^{\frac{\gamma(p-1)^2+2p-2}{p}}} \leq C \left[\int_{\{u_n \geq k\}} f^{\frac{\gamma(p-1)+2}{p}} \right]^{\frac{p}{\gamma(p-1)+2}},$$

which is (17). Then, again by Hölder's inequality, and by (17) and (21),

$$\begin{aligned} \int_{\{u_n \geq k\}} |\nabla u_n| &= \int_{\{u_n \geq k\}} \frac{|\nabla u_n|}{(1+u_n)^{\frac{\gamma(p-1)^2+2(p-1)}{p^2}}} (1+u_n)^{\frac{\gamma(p-1)^2+2(p-1)}{p^2}} \\ &\leq \left[\int_{\{u_n \geq k\}} \frac{|\nabla u_n|^p}{(1+u_n)^{\frac{\gamma(p-1)^2+2p-2}{p}}} \right]^{\frac{1}{p}} \left[\int_{\{u_n \geq k\}} (1+u_n)^{\frac{\gamma(p-1)+2}{p}} \right]^{\frac{p-1}{p}} \\ &\leq C \left[\int_{\{u_n \geq k\}} f^{\frac{\gamma(p-1)+2}{p}} \right]^{\frac{1}{\gamma(p-1)+2}}, \end{aligned} \tag{22}$$

so that (18) is proved. Finally, follows from (13) we get,

$$\alpha \int_{\Omega} \frac{|\nabla T_k(u_n)|^p}{(1+u_n)^{\gamma(p-1)}} + \int_{\Omega} u_n T_k(u_n) \leq \int_{\Omega} f_n T_k(u_n),$$

aftering dropping the second nonnegative term, we get

$$\alpha \int_{\Omega} |\nabla T_k(u_n)|^p \leq k(1+k)^{\gamma(p-1)} \int_{\Omega} f,$$

which is (19).

Lemma 3. *If $\{u_n\}$ is the sequence of solutions to (15), there exists a subsequence, still denoted by $\{u_n\}$, and a function u in $L^{\frac{\gamma(p-1)+2}{p}}(\Omega)$, with $T_k(u)$ belonging to $W_0^{1,p}(\Omega)$ for every $k > 0$, such that u_n almost everywhere converges to u in Ω , and $T_k(u_n)$ weakly converges to $T_k(u)$ in $W_0^{1,p}(\Omega)$.*

Proof. Consider (17) written for $k = 0$:

$$\int_{\Omega} \frac{|\nabla u_n|^p}{(1+u_n)^{\frac{\gamma(p-1)^2+2p-2}{p}}} \leq C \left[\int_{\Omega} f^{\frac{\gamma(p-1)+2}{p}} \right]^{\frac{p}{\gamma(p-1)+2}}. \quad (23)$$

Since (if $\gamma \neq \frac{(p-1)^2+1}{(p-1)^2}$),

$$\frac{|\nabla u_n|^p}{(1+u_n)^{\frac{\gamma(p-1)^2+2p-2}{p}}} = \left| \frac{p^2}{p^2 - \gamma(p-1)^2 - 2(p-1)} \right|^p \left| \nabla \left[(1+u_n)^{1 - \frac{\gamma(p-1)^2+2(p-1)}{p^2}} - 1 \right] \right|^p,$$

the sequence $v_n = \frac{p^2}{p^2 - \gamma(p-1)^2 - 2(p-1)} \left[(1+u_n)^{1 - \frac{\gamma(p-1)^2+2(p-1)}{p^2}} - 1 \right]$ is bounded in $W_0^{1,p}(\Omega)$ by (23). If $\gamma = \frac{(p-1)^2+1}{(p-1)^2}$, we have

$$\frac{|\nabla u_n|^p}{(1+u_n)^p} = |\nabla \log(1+u_n)|^p,$$

so that $v_n = \log(1+u_n)$ is bounded in $W_0^{1,p}(\Omega)$. In both cases, up to a subsequence still denoted by v_n , v_n converges to some function v weakly in $W_0^{1,p}(\Omega)$, strongly in $L^q(\Omega)$ according to Rellich-Kondrachov theorem ($q = \frac{np}{n-p}$), and almost everywhere in Ω . If $\gamma < \frac{(p-1)^2+1}{(p-1)^2}$, define

$$u(x) = \left[\frac{p^2 - \gamma(p-1)^2 - 2(p-1)}{p^2} v + 1 \right]^{\frac{p^2}{p^2 - \gamma(p-1)^2 - 2(p-1)}} - 1,$$

if $\gamma > \frac{(p-1)^2+1}{(p-1)^2}$ define

$$u(x) = \begin{cases} \left[\frac{p^2 - \gamma(p-1)^2 - 2(p-1)}{p^2} v + 1 \right]^{\frac{p^2}{p^2 - \gamma(p-1)^2 - 2(p-1)}} - 1 & \text{if } v(x) < \frac{p^2}{2(p-1) + \gamma(p-1)^2 - p^2}, \\ +\infty & \text{if } v(x) = \frac{p^2}{2(p-1) + \gamma(p-1)^2 - p^2}, \end{cases}$$

while if $\gamma = \frac{(p-1)^2+1}{(p-1)^2}$, define

$$u(x) = e^{v(x)} - 1.$$

Thus, u_n almost everywhere converges, up to a subsequence still denoted by u_n , to u . From now on, we will still consider the particular subsequence, for which it holds that u_n almost everywhere converges to u .

We now use (16) written for $k = 0$:

$$\int_{\Omega} u_n^{\frac{\gamma(p-1)+2}{p}} \leq \int_{\Omega} f^{\frac{\gamma(p-1)+2}{p}} \leq C.$$

Since u_n almost everywhere converges to u , we have from Fatou's lemma that

$$\int_{\Omega} u^{\frac{\gamma(p-1)+2}{p}} \leq C.$$

Hence u belongs to $L^{\frac{\gamma(p-1)+2}{p}}(\Omega)$, which implies that u is almost everywhere finite (note that if $\gamma > \frac{(p-1)^2+1}{(p-1)^2}$ this fact did not follow from the definition of u , since v could have assumed the value $\frac{p^2}{2(p-1)+\gamma(p-1)^2-p^2}$ on a set of positive measure).

Let now $k > 0$; since from (19) it follows that the sequence $\{T_k(u_n)\}$ is bounded in $W_0^{1,p}(\Omega)$, there exists a subsequence $T_k(u_{n_j})$ which weakly converges to some function v_k in $W_0^{1,p}(\Omega)$. Using the almost everywhere convergence of u_n to u , we have that $v_k = T_k(u)$. Since the limit is independent on the subsequence, then the whole sequence $\{T_k(u_n)\}$ weakly converges to $T_k(u)$, for every $k > 0$.

Remark 1. Using the fact that $T_k(u)$ is in $W_0^{1,p}(\Omega)$ for every $k > 0$, and the results of [11], we have that there exists a unique measurable function v with values in $\mathbb{R}^{\mathbb{N}}$, such that

$$\nabla T_k(u) = v\chi_{|u|\leq k} \quad \text{almost everywhere in } \Omega, \text{ for every } k > 0.$$

Following again [11], We will define $\nabla u = v$, the approximate gradient of u .

Our next result deals with the strong convergence of $T_k(u_n)$ in $W_0^{1,p}(\Omega)$.

Proposition 1. *Let u_n and u be the sequence of solutions to problems (15) and the function in $L^{\frac{\gamma(p-1)+2}{p}}(\Omega)$ given by Lemma 3. Then, for every fixed $k > 0$, $T_k(u_n)$ strongly converges to $T_k(u)$ in $W_0^{1,p}(\Omega)$, as n tends to infinity.*

Proof. We follow the proof of [6]. Let $h > k$ and choose $T_{2k}[u_n - T_h(u_n) + T_k(u_n) - T_k(u)]$ as test function in (15). We have

$$\begin{aligned} & \int_{\Omega} \frac{a(x, \nabla u_n) \cdot \nabla T_{2k}[u_n - T_h(u_n) + T_k(u_n) - T_k(u)]}{(1 + u_n)^{\gamma(p-1)}} \\ &= - \int_{\Omega} (u_n - f_n) T_{2k}[u_n - T_h(u_n) + T_k(u_n) - T_k(u)]. \end{aligned} \quad (24)$$

We observe that the right hand side converges to zero as first n and then h tend to infinity, since u_n converges to u almost everywhere in Ω and u_n and f_n are bounded in $L^{\frac{\gamma(p-1)+2}{p}}(\Omega)$. Thus, if we define $\varepsilon(n, h)$ as any quantity such that

$$\lim_{h \rightarrow +\infty} \lim_{n \rightarrow +\infty} \varepsilon(n, h) = 0.$$

Then

$$\int_{\Omega} (u_n - f_n) T_{2k}[u_n - T_h(u_n) + T_k(u_n) - T_k(u)] = \varepsilon(n, h).$$

Let $M = k + h$. Observing that $\nabla T_{2k}[u_n - T_h(u_n) + T_k(u_n) - T_k(u)] = 0$ if $u_n \geq M$, by (24) we have

$$\begin{aligned} \varepsilon(n, h) &= \int_{\{u_n \leq k\}} \frac{a(x, \nabla T_M(u_n)) \cdot \nabla [u_n - T_h(u_n) + T_k(u_n) - T_k(u)]}{(1 + u_n)^{\gamma(p-1)}} \\ &+ \int_{\{u_n \geq k\}} \frac{a(x, \nabla T_M(u_n)) \cdot \nabla [u_n - T_h(u_n) + T_k(u_n) - T_k(u)]}{(1 + u_n)^{\gamma(p-1)}}. \end{aligned}$$

Since $u_n - T_h(u_n) = 0$ in $\{u_n \leq k\}$ and $\nabla T_k(u_n) = 0$ in $\{u_n \geq k\}$, we have, using that $a(x, 0) = 0$,

$$\begin{aligned} \varepsilon(n, h) = & \int_{\Omega} \frac{a(x, \nabla T_k(u_n)) \cdot \nabla [T_k(u_n) - T_k(u)]}{(1 + u_n)^{\gamma(p-1)}} \\ & + \int_{\{u_n \geq k\}} \frac{a(x, \nabla T_M(u_n)) \cdot \nabla [u_n - T_h(u_n)]}{(1 + u_n)^{\gamma(p-1)}} \\ & - \int_{\{u_n \geq k\}} \frac{a(x, \nabla T_M(u_n)) \cdot \nabla T_k(u)}{(1 + u_n)^{\gamma(p-1)}}. \end{aligned}$$

Dropping the second positive term of the right hand side, and subtracting and adding a term

$$\int_{\Omega} \frac{a(x, \nabla T_k(u))}{(1 + u_n)^{\gamma(p-1)}},$$

so that

$$\begin{aligned} \varepsilon(n, h) \geq & \int_{\Omega} \frac{[a(x, \nabla T_k(u_n)) - a(x, \nabla T_k(u))] \cdot \nabla [T_k(u_n) - T_k(u)]}{(1 + k)^{\gamma(p-1)}} \\ & + \int_{\Omega} \frac{a(x, \nabla T_k(u)) \cdot \nabla [T_k(u_n) - T_k(u)]}{(1 + u_n)^{\gamma(p-1)}} \\ & - \int_{\{u_n \geq k\}} \frac{a(x, \nabla T_M(u_n)) \cdot \nabla T_k(u)}{(1 + u_n)^{\gamma(p-1)}} = I_n + J_n - K_n. \end{aligned}$$

The last two terms tend to zero as n tends to infinity. Indeed

$$\lim_{n \rightarrow \infty} J_n = \lim_{n \rightarrow \infty} \int_{\Omega} \frac{a(x, \nabla T_k(u)) \cdot \nabla [T_k(u_n) - T_k(u)]}{(1 + u_n)^{\gamma(p-1)}} = 0.$$

Since $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega)$ and u_n converges almost everywhere, it follows that $\nabla [T_k(u_n) - T_k(u)]$ weakly converges to zero. The numerator $a(x, \nabla T_k(u)) \cdot \nabla [T_k(u_n) - T_k(u)]$ is controlled by a bounded constant, and $(1 + u_n)^{\gamma(p-1)}$ remains bounded as $n \rightarrow \infty$. Since the integrand converges almost everywhere to zero, and combining the boundedness and convergence analysis, the limit of the integral is zero by applying the Lebesgue Dominated Convergence Theorem. The last term can be rewritten as

$$K_n = \int_{\Omega} \frac{a(x, \nabla T_M(u_n)) \cdot \nabla T_k(u) \chi_{\{u_n \geq k\}}}{(1 + u_n)^{\gamma(p-1)}}.$$

Since M is fixed with respect to n , then the sequence $\{a(x, \nabla T_M(u_{n_j}))\}$ is bounded in $(L^p(\Omega))^N$. Hence, there exists σ in $(L^p(\Omega))^N$, and a subsequence $\{a(x, \nabla T_M(u_{n_j}))\}$, such that

$$\lim_{j \rightarrow +\infty} a(x, \nabla T_M(u_{n_j})) = \sigma,$$

weakly in $(L^p(\Omega))^N$. On the other hand,

$$\lim_{n \rightarrow +\infty} \frac{\nabla T_k(u) \chi_{\{u_n \geq k\}}}{(1 + u_n)^{\gamma(p-1)}} = \frac{\nabla T_k(u) \chi_{\{u \geq k\}}}{(1 + u)^{\gamma(p-1)}} = 0,$$

strongly in $(L^{p'}(\Omega))^N$, and so

$$\lim_{j \rightarrow +\infty} K_{n_j} = \lim_{j \rightarrow +\infty} \int_{\Omega} \frac{a(x, \nabla T_M(u_{n_j})) \cdot \nabla T_k(u) \chi_{\{u_{n_j} \geq k\}}}{(1 + u_{n_j})^{\gamma(p-1)}} = 0.$$

Since the limit does not depend on the subsequence, we have

$$\lim_{n \rightarrow +\infty} K_n = \lim_{n \rightarrow +\infty} \int_{\Omega} \frac{a(x, \nabla T_M(u_n)) \cdot \nabla T_k(u) \chi_{\{u_n \geq k\}}}{(1 + u_n)^{\gamma(p-1)}} = 0.$$

Therefore,

$$\varepsilon(n, h) \geq I_n = \int_{\Omega} \frac{[a(x, \nabla T_k(u_n)) - a(x, \nabla T_k(u))] \cdot \nabla [T_k(u_n) - T_k(u)]}{(1 + k)^{\gamma(p-1)}},$$

so that, thanks to (6),

$$\lim_{n \rightarrow +\infty} [a(x, \nabla T_k(u_n)) - a(x, \nabla T_k(u))] \cdot \nabla [T_k(u_n) - T_k(u)] = 0.$$

Using this formula, (6) and the results of [12], we then conclude that $T_k(u_n)$ strongly converges to $T_k(u)$ in $W_0^{1,p}(\Omega)$, as desired.

Corollary 1. *Let u_n and u be as in Proposition 2.8. Then ∇u_n converges to ∇u almost everywhere in Ω , where ∇u has defined in Remark 3.1.*

Lemma 4. *Let u_n and u be as in Proposition 2.8. Then ∇u_n strongly converges to ∇u in $(L^1(\Omega))^N$. Moreover u_n strongly converges to u in $L^{\frac{\gamma(p-1)+2}{p}}(\Omega)$.*

Proof. We begin by proving the convergence of ∇u_n to ∇u . Let $\varepsilon > 0$, and let $k > 0$ be sufficiently large such that

$$\left[\int_{\{u_n \geq k\}} f^{\frac{\gamma(p-1)+2}{p}} \right]^{\frac{p-1}{\gamma(p-1)+2}} < \varepsilon, \quad (25)$$

uniformly with respect to n . This can be done thanks to (21) and to the absolute continuity of the integral. Let E be a measurable set, writing

$$\int_E |\nabla u_n| = \int_E |\nabla T_k(u_n)| + \int_{E \cap \{u_n \geq k\}} |\nabla u_n|.$$

We have, by (18), and by (25),

$$\int_E |\nabla u_n| \leq \int_E |\nabla T_k(u_n)| + C\varepsilon.$$

Using Hölder inequality and (19), we obtain

$$\int_E |\nabla u_n| \leq C \text{meas}(E)^{\frac{p-1}{p}} k^{\frac{1}{p}} (1 + k)^{\frac{\gamma(p-1)}{p}} \left(\int_{\Omega} f \right)^{\frac{1}{p}} + C\varepsilon.$$

Choosing $\text{meas}(E)$ small enough (recall that k is now fixed) we have

$$\int_E |\nabla u_n| \leq C\varepsilon,$$

uniformly with respect to n , where C does not depend on n or ε . Since ∇u_n almost everywhere converges to ∇u by Corollary 3.1, we can apply Vitali's theorem to obtain the strong convergence of ∇u_n to ∇u in $(L^1(\Omega))^N$. As for the second convergence, by (16) we have

$$\begin{aligned} \int_E u_n^{\frac{\gamma(p-1)+2}{p}} &\leq \int_{E \cap \{u_n \leq k\}} u_n^{\frac{\gamma(p-1)+2}{p}} + \int_{E \cap \{u_n \geq k\}} u_n^{\frac{\gamma(p-1)+2}{p}} \\ &\leq k^{\frac{\gamma(p-1)+2}{p}} |E| + \int_{\{u_n \geq k\}} f^{\frac{\gamma(p-1)+2}{p}}. \end{aligned}$$

As before, we first choose k such that the second integral is small, uniformly with respect to n , and then the measure of E small enough such that the first term is small. The almost everywhere convergence of u_n to u , and Vitali's theorem, then imply that u_n strongly converges to u in $L^{\frac{\gamma(p-1)+2}{p}}(\Omega)$.

Remark 2. Since we have proved that ∇u_n strongly converges to ∇u in $(L^1(\Omega))^N$, so that u belongs to $W_0^{1,1}(\Omega)$, then the approximate gradient ∇u of u is nothing but the distributional gradient of u .

Now we will prove Theorem 1.1.

Proof. Using the previous estimates, we pass to the limit as n tends to infinity, in the weak formulation of (15). Starting from

$$\int_{\Omega} \frac{a(x, \nabla u_n) \cdot \nabla \psi}{(1 + u_n)^{\gamma(p-1)}} + \int_{\Omega} u_n \psi = \int_{\Omega} f_n \psi, \quad \forall \psi \in W_0^{1,\infty}(\Omega).$$

Recall Lemma 3.3 and the definition of f_n , the sequences $\{u_n\}$ and $\{f_n\}$ strongly converge to u and f in $L^{\frac{\gamma(p-1)+2}{p}}(\Omega)$, hence in $L^1(\Omega)$. For the first integral, we have that $a(x, \nabla u_n)$ converges almost everywhere in Ω to $a(x, \nabla u)$ thanks to Corollary 3.1, and the continuity assumption on $a(x, \cdot)$; furthermore, (5) implies that

$$|a(x, \nabla u_n)| \leq \beta |\nabla u_n|^{p-1},$$

and the right hand side is compact in $L^1(\Omega)$ by Lemma 3.3. Thus, by Vitali's theorem $a(x, \nabla u_n)$ strongly converges to $a(x, \nabla u)$ in $(L^1(\Omega))^N$, so that

$$\lim_{n \rightarrow +\infty} \int_{\Omega} \frac{a(x, \nabla u_n) \cdot \nabla \varphi}{(1 + |u_n|)^{\gamma(p-1)}} = \int_{\Omega} \frac{a(x, \nabla u) \cdot \nabla \varphi}{(1 + |u|)^{\gamma(p-1)}},$$

where we have also used that u_n almost everywhere converges to u , and Lebesgue's theorem. Thus, we have that

$$\int_{\Omega} \frac{a(x, \nabla u) \cdot \nabla \varphi}{(1 + |u|)^{\gamma(p-1)}} + \int_{\Omega} u \varphi = \int_{\Omega} f \varphi, \quad \forall \varphi \in W_0^{1,\infty}(\Omega),$$

4. Uniqueness of the solution obtained by approximation

Let $f \in L^{\frac{\gamma(p-1)+2}{p}}(\Omega)$, let f_n be a sequence of $L^\infty(\Omega)$ functions converging to f in $L^{\frac{\gamma(p-1)+2}{p}}(\Omega)$, with $0 \leq f_n \leq f$, and let u_n be a solution of (15). In Sect.3 we proved the existence of a distributional solution u in $W_0^{1,1}(\Omega) \cap L^{\frac{\gamma(p-1)+2}{p}}(\Omega)$ to (3), such that, up to a subsequence,

$$\lim_{n \rightarrow \infty} \|u_n - u\|_{W_0^{1,1}(\Omega) \cap L^{\frac{\gamma(p-1)+2}{p}}(\Omega)} = 0. \quad (26)$$

Now, let $g \in L^{\frac{\gamma(p-1)+2}{p}}(\Omega)$, let g_n be a sequence of $L^\infty(\Omega)$ functions converging to g in $L^{\frac{\gamma(p-1)+2}{p}}(\Omega)$, with $0 \leq g_n \leq g$, and let $z_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a weak solution of

$$\begin{cases} -\operatorname{div} \left(\frac{a(x, \nabla z_n)}{(1+z_n)^{\gamma(p-1)}} \right) + z_n = g_n & \text{in } \Omega, \\ z_n = 0 & \text{on } \partial\Omega. \end{cases} \quad (27)$$

Then, up to a subsequence, we can assume that

$$\lim_{n \rightarrow \infty} \|z_n - z\|_{W_0^{1,1}(\Omega) \cap L^{\frac{\gamma(p-1)+2}{p}}(\Omega)} = 0, \quad (28)$$

where z in $W_0^{1,1}(\Omega) \cap L^{\frac{\gamma(p-1)+2}{p}}(\Omega)$ is a distributional solution of

$$\begin{cases} -\operatorname{div} \left(\frac{a(x, \nabla z)}{(1+z)^{\gamma(p-1)}} \right) + z = f & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \quad (29)$$

Our result, which will imply the uniqueness of the solution by approximation (see [13]) of (3), is the following.

Theorem 2. Assume that u_n and z_n are solutions of (15) and (27) respectively, and that (26) and (29) hold true, with u and z solutions of (3) and (28) respectively. Then

$$\int_{\Omega} |u - z| \leq \int_{\Omega} |f - g|. \quad (30)$$

Moreover,

$$f \leq g \text{ a.e. in } \Omega \text{ implies } u \leq z \text{ a.e. in } \Omega. \quad (31)$$

Proof. Subtracting (27) from (15) we obtain

$$-\operatorname{div} \left(\frac{a(x, \nabla u_n)}{(1+u_n)^{\gamma(p-1)}} - \frac{a(x, \nabla z_n)}{(1+z_n)^{\gamma(p-1)}} \right) + u_n - z_n = f_n - g_n.$$

Choosing $T_h(u_n - z_n)$ as a test function we have

$$\begin{aligned} & \int_{\Omega} \left[\frac{a(x, \nabla u_n)}{(1+u_n)^{\gamma(p-1)}} - \frac{a(x, \nabla z_n)}{(1+z_n)^{\gamma(p-1)}} \right] \cdot \nabla T_h(u_n - z_n) \\ & + \int_{\Omega} (u_n - z_n) T_h(u_n - z_n) = \int_{\Omega} (f_n - g_n) T_h(u_n - z_n). \end{aligned}$$

This equality can be written in an equivalent way as

$$\begin{aligned} & \int_{\Omega} \frac{[a(x, \nabla u_n) - a(x, \nabla z_n)] \cdot \nabla T_h(u_n - z_n)}{(1 + |u_n|)^{\gamma(p-1)}} \\ & + \int_{\Omega} (u_n - z_n) T_h(u_n - z_n) = \int_{\Omega} (f_n - g_n) T_h(u_n - z_n) \\ & - \int_{\Omega} \left[\frac{1}{(1 + u_n)^{\gamma(p-1)}} - \frac{1}{(1 + z_n)^{\gamma(p-1)}} \right] a(x, \nabla z_n) \cdot \nabla T_h(u_n - z_n). \end{aligned}$$

By (4), the first term of the left hand side is nonnegative, so that it can be dropped; using Lagrange's theorem on the last term of the right hand side, we therefore have, since the absolute value of the derivative of the function $s \mapsto \frac{1}{(1+s)^{\gamma(p-1)}}$ is bounded by $\gamma(p-1)$,

$$\int_{\Omega} (u_n - z_n) T_h(u_n - z_n) \leq \int_{\Omega} (f_n - g_n) T_h(u_n - z_n) + (\gamma - 1) h \int_{\Omega} |a(x, \nabla z_n)| |\nabla T_h(u_n - z_n)|.$$

Dividing by h we obtain

$$\int_{\Omega} (u_n - z_n) \frac{T_h(u_n - z_n)}{h} \leq \int_{\Omega} |f_n - g_n| \left| \frac{T_h(u_n - z_n)}{h} \right| + (\gamma - 1) \int_{\Omega} |a(x, \nabla z_n)| |\nabla T_h(u_n - z_n)|.$$

Since, for every fixed n , u_n and z_n belong to $W_0^{1,p}(\Omega)$, and $a(x, \xi)$ satisfies (5), the limit as h tends to zero gives

$$\int_{\Omega} |u_n - z_n| \leq \int_{\Omega} |f_n - g_n|, \quad (32)$$

which then yields (30) passing to the limit and using the second part of Lemma 3.3.

The use of $T_h(u_n - z_n)^+$ as a test function and the same technique as above imply that

$$\int_{\Omega} (u_n - z_n)^+ \leq \int_{\{u_n \geq z_n\}} (f_n - g_n).$$

Hence, passing to the limit as n tends to infinity, we obtain, if we suppose that $f \leq g$ almost everywhere in Ω ,

$$\int_{\Omega} (u - z)^+ \leq \int_{\{u \geq z\}} (f - g) \leq 0,$$

so that (31) is proved.

Thanks to (30), we can prove that problem (3) has a unique solution obtained by approximation.

Corollary 4.2. *There exists a unique solution obtained by approximation of (3), in the sense that the solution u in $W_0^{1,1}(\Omega) \cap L^{\frac{\gamma(p-1)+2}{p}}(\Omega)$ obtained as limit of the sequence u_n of solutions of (15) does not depend on the sequence f_n chosen to approximate the datum f in $L^{\frac{\gamma(p-1)+2}{p}}(\Omega)$.*

5. Conclusion

In this paper, we have studied the existence, regularity, and uniqueness of distributional solutions for equation (3) and extended the results of Boccardo in [6]. Our analysis demonstrates that for $1 < p < 2$, certain theoretical results of second-order degenerate elliptic equations can be generalized to the p -Laplacian degenerate elliptic equation, thereby further refining the research framework in this field. However, when $p > 2$, due to the poor integrability of ∇u_n , the existing approach fails to yield similar conclusions. This limitation suggests potential directions for future research, where improved analytical techniques may be explored to achieve a more comprehensive understanding of such equations.

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