

# *A New Subclass of H-matrices: $\gamma$ -DZT Matrices*

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**Abstract.** H-matrices have wide applications in numerical analysis, control theory, matrix theory and statistics. An important method to study the properties of H-matrices is to consider its subclasses. H-matrices have wide applications. This paper introduces a new subclass of H-matrices, named  $\gamma$ -DZT matrices. We prove that  $\gamma$ -SDD matrices and DZT matrices belong to  $\gamma$ -DZT matrices. This paper introduces a new matrix subclass:  $\gamma$ -DZT matrices. By constructing the scaling matrix, we prove that the class of  $\gamma$ -DZT matrices belongs to H-matrices. Moreover,  $\gamma$ -SDD matrices and DZT matrices belong to  $\gamma$ -DZT matrices. This paper successfully introduces a new subclass of matrices, the  $\gamma$ -DZT matrix, and proves that the  $\gamma$ -DZT matrix belongs to the H-matrix by constructing a scaling matrix. The inclusion relationships among the four types of matrices have been clarified, and H-matrices is clarified. These conclusions provide new theoretical basis and research directions for further studies on the properties and applications of H-matrix, and are of great significance in the development of matrix theory and related application fields.

**Keywords:**  $\gamma$ -DZT matrix, DZT matrix, Scaling matrix

## 1. Introduction

H-matrices have wide applications in numerical analysis, control theory, matrix theory and statistics [1-4]. An important method to study the properties of H-matrices is to consider its subclasses. H-matrices include many well-known subclasses, such as strictly diagonally dominant (SDD) matrices, doubly strictly diagonally dominant (DSDD) matrices,  $\Sigma$ -SDD matrices,  $\gamma$ -SDD matrices, weakly chained diagonally dominant matrices, Nekrasov matrices,  $\Sigma$ -Nekrasov matrices,

Dashnic-Zusmanovich (DZ) matrices and eventually SDD matrices.

The class of Dashnic-Zusmanovich type (DZT) matrices were introduced by Zhao et al. in 2018 [5]. They proved that DZT matrices belong to the class of H-matrices. Many results about DZT matrices have been obtained. One can refer to [6-9].

This paper introduces a new matrix subclass:  $\gamma$ -DZT matrices. By constructing the scaling matrix, we prove that the class of  $\gamma$ -DZT matrices belongs to H-matrices. Moreover,  $\gamma$ -SDD matrices and DZT matrices belong to  $\gamma$ -DZT matrices.

## 2. Preliminaries

We use  $C^{n \times n}$  denote all the complex matrices with the order  $n$ , and denote  $\langle n \rangle = \{1, 2, \dots, n\}$ . For any  $A \in C^{n \times n}$ , define:

$$r_i(A) = \sum_{j \neq i}^n |a_{ij}| \quad (2.1)$$

$$c_i(A) = \sum_{j \neq i}^n |a_{ji}| \quad (2.2)$$

$$r_i^S(A) = \sum_{j \neq i, j \in S}^n |a_{ij}| \quad (S \subseteq \langle n \rangle) \quad (2.3)$$

$$c_i^S(A) = \sum_{j \neq i, j \in S}^n |a_{ji}| \quad (S \subseteq \langle n \rangle) \quad (2.4)$$

$$\Gamma_i(A) = \left\{ j \in \langle n \rangle - \{i\} : \left( |a_{ii}| - r_i^{\langle n \rangle - \{j\}}(A) \right) |a_{jj}| > |a_{ij}| r_j(A) \right\}, i \in \langle n \rangle \quad (2.5)$$

If  $S = \langle n \rangle - \{j\}$ ,

$r_i^S(A)$  and  $c_i^S(A)$  are simplified as  $r_i^{\{j\}}(A)$  and  $c_i^{\{j\}}(A)$ , respectively.

**Definition 2.1.** Let  $A = (a_{ij}) \in C^{n \times n}$ , We say that  $A$  is an SDD matrix if for all  $i \in \langle n \rangle$ , it holds that  $|a_{ii}| > r_i(A)$ .

**Definition 2.2.** Let  $A = (a_{ij}) \in C^{n \times n}$ , if there exists  $X = \text{diag}(x_1 x_2 \dots x_n)$  such that  $AX$  is an SDD matrix, we call  $A$  is a nonsingular H-matrix.  $X$  is called scaling matrix of  $A$ .

**Definition 2.3.** Let  $A = (a_{ij}) \in C^{n \times n}$ , if for all  $i \in \langle n \rangle$ , either  $i \in N^+(A)$ , or  $\Gamma_i(A) \neq \emptyset$ , we call  $A$  is a DZT matrix.

**Definition 2.4.** Let  $A = (a_{ij}) \in C^{n \times n}$ , we say that  $A$  is a  $\gamma$ -SDD matrix if there exists an  $\alpha \in [0, 1]$  such that  $A$  is an  $\alpha$ -SDD matrix, i.e.,

$$|a_{ii}| > \alpha r_i(A) + (1 - \alpha) c_i(A), \forall i \in \langle n \rangle.$$

## 3. Main results

**Definition 3.1.** A matrix  $A = [a_{ij}] \in C^{n \times n}$ ,  $n \geq 2$ , is called an  $\alpha$ -DZT matrix if there exists  $\alpha \in [0, 1]$ , such that for all  $i \in \langle n \rangle$ , there exists  $j \neq i$  satisfying

$$\left( |a_{ii}| - \alpha \cdot r_i^{\{j\}}(A) - (1 - \alpha) c_i^{\{j\}}(A) \right) |a_{jj}| > \left( \alpha |a_{ij}| + (1 - \alpha) |a_{ji}| \right) \left( \alpha \cdot r_j(A) - (1 - \alpha) c_j(A) \right), \quad (3.1)$$

We say that  $A$  is a  $\gamma$ -DZT matrix if there exists an  $\alpha \in [0, 1]$  such that  $A$  is an  $\alpha$ -DZT matrix. It is easy to see that

$$|a_{ii}| > \frac{\alpha \cdot r_j(A) + (1 - \alpha) c_j(A)}{|a_{jj}|} \left( \alpha |a_{ij}| + (1 - \alpha) |a_{ji}| \right) + \alpha \cdot r_i^{\{j\}}(A) + (1 - \alpha) c_i^{\{j\}}(A). \quad (3.2)$$

Let

$$N_1 = \{i : |a_{ii}| \leq \alpha \cdot r_i(A) + (1 - \alpha)c_i(A)\},$$

$$N_2 = \{i : |a_{ii}| > \alpha \cdot r_i(A) + (1 - \alpha)c_i(A)\}.$$

Theorem 3.1. Let  $A = [a_{ij}] \in C^{n \times n}, n \geq 2$  be an  $\alpha - DZT$  matrix, there exists a diagonal matrix  $D = \text{diag}(d_1, d_2, \dots, d_n)$  such that  $DAD$  is an SDD matrix, where

$$d_i = \begin{cases} 1, i \in N_1 \\ \frac{\alpha \cdot r_i(A) + (1 - \alpha)c_i(A)}{|a_{ii}|} + \varepsilon, i \in N_2 \end{cases}$$

Proof. Let  $B = DAD$ . We will show that  $|b_{ii}| > \alpha \cdot r_i(B) + (1 - \alpha)c_i(B)$  for all  $i \in \langle n \rangle$ .

Case 1.  $i \in N_1$ . There must be  $j_0 \in N_2$ , that is  $\frac{\alpha \cdot r_{j_0}(A) + (1 - \alpha)c_{j_0}(A)}{|a_{j_0 j_0}|} < 1$ , and  $d_i = 1$ , then  $|b_{ii}| = |a_{ii}|$ .

$$\begin{aligned} r_i(B) &= \sum_{j \neq i} |b_{ij}| = \sum_{j \in N_1} |a_{ij}| + \sum_{j \in N_2} |a_{ij}| \cdot d_j \\ &= \sum_{j \in N_1} |a_{ij}| + \sum_{j \in N_2} |a_{ij}| \cdot \frac{\alpha \cdot r_j(A) + (1 - \alpha)c_j(A)}{|a_{jj}|} \end{aligned}$$

$$\begin{aligned} \alpha \cdot r_i(B) &= \alpha \sum_{j \in N_1} |a_{ij}| + \alpha \sum_{j \in N_2} |a_{ij}| \cdot \frac{\alpha \cdot r_j(A) + (1 - \alpha)c_j(A)}{|a_{jj}|} \\ &= \alpha \sum_{j \in N_1} |a_{ij}| + \alpha \sum_{j \in N_2 \setminus \{j_0\}} |a_{ij}| \cdot \frac{\alpha \cdot r_j(A) + (1 - \alpha)c_j(A)}{|a_{jj}|} + \alpha |a_{ij_0}| \cdot \frac{\alpha \cdot r_{j_0}(A) + (1 - \alpha)c_{j_0}(A)}{|a_{j_0 j_0}|} \\ &\leq \alpha \sum_{j \in N_1} |a_{ij}| + \alpha \sum_{j \in N_2 \setminus \{j_0\}} |a_{ij}| + \alpha |a_{ij_0}| \cdot \frac{\alpha \cdot r_{j_0}(A) + (1 - \alpha)c_{j_0}(A)}{|a_{j_0 j_0}|} \end{aligned}$$

Because

$$\alpha \sum_{j \in N_1} |a_{ij}| + \alpha \sum_{j \in N_2 \setminus \{j_0\}} |a_{ij}| = \alpha \cdot r_i^{\{j_0\}}(A),$$

then

$$\alpha \cdot r_i(B) < \alpha \cdot r_i^{\{j_0\}}(A) + \alpha |a_{ij_0}| \cdot \frac{\alpha \cdot r_{j_0}(A) + (1 - \alpha)c_{j_0}(A)}{|a_{j_0 j_0}|}.$$

$$\begin{aligned} c_i(B) &= \sum_{j \neq i} |b_{ji}| = \sum_{j \in N_1} |a_{ji}| + d_i \sum_{j \in N_2} |a_{ji}| \\ &= \sum_{j \in N_1} |a_{ji}| + \frac{\alpha \cdot r_j(A) + (1 - \alpha)c_j(A)}{|a_{jj}|} \sum_{j \in N_2} |a_{ji}| \end{aligned}$$

$$\begin{aligned} (1 - \alpha) \cdot c_i(B) &= (1 - \alpha) \sum_{j \in N_1} |a_{ji}| + (1 - \alpha) \cdot \frac{\alpha \cdot r_j(A) + (1 - \alpha)c_j(A)}{|a_{jj}|} \sum_{j \in N_2} |a_{ji}| \\ &= (1 - \alpha) \sum_{j \in N_1} |a_{ji}| + (1 - \alpha) \cdot \frac{\alpha \cdot r_j(A) + (1 - \alpha)c_j(A)}{|a_{jj}|} \sum_{j \in N_2 \setminus \{j_0\}} |a_{ji}| \\ &\quad + (1 - \alpha) |a_{j_0 i}| \cdot \frac{\alpha \cdot r_{j_0}(A) + (1 - \alpha)c_{j_0}(A)}{|a_{j_0 j_0}|} \\ &\leq (1 - \alpha) \sum_{j \in N_1} |a_{ji}| + (1 - \alpha) \sum_{j \in N_2 \setminus \{j_0\}} |a_{ji}| + (1 - \alpha) |a_{j_0 i}| \cdot \frac{\alpha \cdot r_{j_0}(A) + (1 - \alpha)c_{j_0}(A)}{|a_{j_0 j_0}|} \end{aligned}$$

Because

$$(1 - \alpha) \sum_{j \in N_1} |a_{ji}| + (1 - \alpha) \sum_{j \in N_2 \setminus \{j_0\}} |a_{ji}| = (1 - \alpha) \cdot c_i^{\{j_0\}}(A)$$

,

then

$$(1 - \alpha) \cdot c_i(B) < (1 - \alpha) \cdot c_i^{\{j_0\}}(A) + (1 - \alpha) |a_{j_0 i}| \cdot \frac{\alpha \cdot r_{j_0}(A) + (1 - \alpha) c_{j_0}(A)}{|a_{j_0 j_0}|},$$

and because of formula (3.2),

we proved  $|b_{ii}| > \alpha \cdot r_i(B) + (1 - \alpha) c_i(B)$ .

Case 2.  $i \in N_2$ . There must be  $\frac{\alpha \cdot r_i(A) + (1 - \alpha) c_i(A)}{|a_{ii}|} < 1$ ,

and  $d_i = \frac{\alpha \cdot r_i(A) + (1 - \alpha) c_i(A)}{|a_{ii}|} + \varepsilon$ , then  $|b_{ii}| = d_i^2 |a_{ii}|$ .

The above formula can be expanded to get

$$\begin{aligned} |b_{ii}| &= \left( \alpha \cdot r_i(A) + (1 - \alpha) c_i(A) + \varepsilon |a_{ii}| \right) \left( \frac{\alpha \cdot r_i(A) + (1 - \alpha) c_i(A)}{|a_{ii}|} + \varepsilon \right) \\ &= \left( \frac{\alpha \cdot r_i(A) + (1 - \alpha) c_i(A)}{|a_{ii}|} \right) \left( \alpha \cdot r_i(A) + \alpha \varepsilon |a_{ii}| \right) + \left( \frac{\alpha \cdot r_i(A) + (1 - \alpha) c_i(A)}{|a_{ii}|} \right) \left( (1 - \alpha) c_i(A) + (1 - \alpha) \varepsilon |a_{ii}| \right) \\ &\quad + d_i (\alpha \cdot r_i(A) + \alpha \varepsilon |a_{ii}|) + d_i ((1 - \alpha) c_i(A) + (1 - \alpha) \varepsilon |a_{ii}|) \end{aligned}$$

,

$$\begin{aligned} r_i(B) &= \sum_{j \neq i} |b_{ij}| = \sum_{j \in N_1} d_i |a_{ij}| + \sum_{j \in N_2} d_i |a_{ij}| \cdot d_j \\ &= d_i \left( \sum_{j \in N_1} |a_{ij}| + \sum_{j \in N_2} |a_{ij}| \cdot d_j \right) \\ \alpha \cdot r_i(B) &= d_i \alpha \left( \sum_{j \in N_1} |a_{ij}| + \sum_{j \in N_2} |a_{ij}| \cdot d_j \right) \\ &= d_i \alpha \left( \sum_{j \in N_1} |a_{ij}| + \sum_{j \in N_2} |a_{ij}| \left( \frac{\alpha \cdot r_i(A) + (1 - \alpha) c_i(A)}{|a_{ii}|} + \varepsilon \right) \right) \\ &= d_i \alpha \left( \sum_{j \in N_1} |a_{ij}| + \sum_{j \in N_2} |a_{ij}| \frac{\alpha \cdot r_i(A) + (1 - \alpha) c_i(A)}{|a_{ii}|} + \varepsilon \sum_{j \in N_2} |a_{ij}| \right) \\ &\leq d_i \alpha \left( \sum_{j \in N_1} |a_{ij}| + \sum_{j \in N_2} |a_{ij}| + \varepsilon \sum_{j \in N_2} |a_{ij}| \right) \\ &= d_i \left( \alpha \cdot r_i(A) + \alpha \cdot \varepsilon \sum_{j \in N_2} |a_{ij}| \right) \\ &< d_i (\alpha \cdot r_i(A) + \alpha \cdot \varepsilon |a_{ii}|) \end{aligned}$$

And,

$$\begin{aligned} c_i(B) &= \sum_{j \neq i} |b_{ji}| = \sum_{j \in N_1} |a_{ji}| d_i + \sum_{j \in N_2} d_j |a_{ji}| \cdot d_i \\ &= \left( \sum_{j \in N_1} |a_{ji}| + \sum_{j \in N_2} d_j |a_{ji}| \right) \cdot d_i \end{aligned}$$

$$\begin{aligned}
 (1 - \alpha) \cdot c_i(B) &= (1 - \alpha) \left( \sum_{j \in N_1} |a_{ji}| + \sum_{j \in N_2} d_j |a_{ji}| \right) \cdot d_i \\
 &= d_i \left( (1 - \alpha) \left( \sum_{j \in N_1} |a_{ji}| + \sum_{j \in N_2} \left( \frac{\alpha \cdot r_i(A) + (1 - \alpha)c_i(A)}{|a_{ii}|} + \varepsilon \right) |a_{ji}| \right) \right) \\
 &= d_i \left( (1 - \alpha) \left( \sum_{j \in N_1} |a_{ji}| + \sum_{j \in N_2} \frac{\alpha \cdot r_i(A) + (1 - \alpha)c_i(A)}{|a_{ii}|} |a_{ji}| + \varepsilon \sum_{j \in N_2} |a_{ji}| \right) \right) \\
 &\leq d_i (1 - \alpha) \left( \sum_{j \in N_1} |a_{ji}| + \sum_{j \in N_2} |a_{ji}| + \varepsilon \sum_{j \in N_2} |a_{ji}| \right) \\
 &= d_i \left( (1 - \alpha)c_i(A) + (1 - \alpha)\varepsilon \sum_{j \in N_2} |a_{ji}| \right) \\
 &< d_i ((1 - \alpha)c_i(A) + (1 - \alpha)\varepsilon |a_{ii}|)
 \end{aligned}$$

Solving simultaneously, we get

$$\alpha \cdot r_i(B) + (1 - \alpha) \cdot c_i(B) < d_i(\alpha \cdot r_i(A) + \alpha \cdot \varepsilon |a_{ii}|) + d_i((1 - \alpha)c_i(A) + (1 - \alpha)\varepsilon |a_{ii}|),$$

then  $|b_{ii}| > \alpha \cdot r_i(B) + (1 - \alpha)c_i(B)$  .we complete the proof.

**Theorem 3.2.** A matrix  $A = [a_{ij}] \in C^{n \times n}, n \geq 2$ , then the matrix A satisfies the following relation:

$$\begin{cases} A \in SDD \Rightarrow A \in DZT \Rightarrow A \in \gamma - DZT \\ A \in SDD \Rightarrow A \in \gamma - SDD \Rightarrow A \in \gamma - DZT \end{cases}$$

**Proof.** If A is a DZT matrix, then A is a 1-DZT matrix. Now we show that if A is an  $\alpha - SDD$  matrix, then A is an  $\alpha - DZT$  matrix.

First of all, by definition, it is obvious that if matrix A belongs to SDD matrix, then A must belong to  $\alpha - SDD$  matrix. In the same way, if matrix A belongs to DZT matrix, then A must belong to the  $\alpha - DZT$  matrix. So now it is only necessary to prove that when A belongs to SDD matrix and  $\alpha - SDD$  matrix, A must belong to DZT matrix and  $\alpha - DZT$  matrix respectively.

By definition, if the hypothesis  $A \in SDD \subseteq A \in DZT$  is true, then

$$\begin{aligned}
 r_i(A) &> \frac{r_{j_0}(A)}{|a_{j_0 j_0}|} |a_{ij_0}| + r_i^{\{j_0\}}(A) \\
 r_i^{\{j_0\}}(A) + |a_{ij_0}| &> \frac{r_{j_0}(A)}{|a_{j_0 j_0}|} |a_{ij_0}| + r_i^{\{j_0\}}(A) \\
 |a_{ij_0}| &> \frac{r_{j_0}(A)}{|a_{j_0 j_0}|} |a_{ij_0}|
 \end{aligned}$$

Because  $\frac{r_{j_0}(A)}{|a_{j_0 j_0}|} < 1$ , therefore, the above inequality is true, that is, the hypothesis is true.

For the argument, we also use the reverse order method, assuming that the conclusion  $A \in \alpha - SDD \subseteq A \in \alpha - DZT$  is true, then by definition,

$$\begin{aligned}
 \alpha \cdot r_i(A) + (1 - \alpha)c_i(A) &> \frac{\alpha \cdot r_{j_0}(A) + (1 - \alpha)c_{j_0}(A)}{|a_{j_0 j_0}|} \left( \alpha |a_{ij_0}| + (1 - \alpha) |a_{j_0 i}| \right) \\
 &\quad + \alpha \cdot r_i^{\{j_0\}}(A) + (1 - \alpha)c_i^{\{j_0\}}(A) \\
 \alpha \cdot \left[ r_i^{\{j_0\}}(A) + |a_{ij_0}| \right] + (1 - \alpha) \left[ c_i^{\{j_0\}}(A) + |a_{j_0 i}| \right] &> \frac{\alpha \cdot r_{j_0}(A) + (1 - \alpha)c_{j_0}(A)}{|a_{j_0 j_0}|} \left( \alpha |a_{ij_0}| + (1 - \alpha) |a_{j_0 i}| \right)
 \end{aligned}$$

$$+\alpha \cdot r_i^{\{j_0\}}(A) + (1-\alpha)c_i^{\{j_0\}}(A)$$

$$\alpha|a_{ij_0}| + (1-\alpha)|a_{j_0i}| > \frac{\alpha r_{j_0}(A) + (1-\alpha)c_{j_0}(A)}{|a_{j_0j_0}|} (\alpha|a_{ij_0}| + (1-\alpha)|a_{j_0i}|)$$

Eliminate the common terms on both sides of the inequality at the same time.

In the same way, because  $\frac{\alpha r_{j_0}(A) + (1-\alpha)c_{j_0}(A)}{|a_{j_0j_0}|} < 1$ , therefore the hypothesis is true. In summary, theorem 2 is proved.

For the proof of theorem 2, we mainly use the core idea of reverse order. We start with the result we want to prove, analyze what conditions are needed to get that result, and then work our way forward to see how we can obtain these conditions from what we already know. Thus, we prove the relationship between SDD, DZT,  $\gamma$ -SDD and  $\gamma$ -DZT matrix.

## 4. Conclusion

This paper successfully introduces a new subclass of matrices, the  $\gamma$ -DZT matrix, and proves that the  $\gamma$ -DZT matrix belongs to the H-matrix by constructing a scaling matrix. The inclusion relationships among the four types of matrices have been clarified, and H-matrices is clarified. These conclusions provide new theoretical basis and research directions for further studies on the properties and applications of H-matrix, and are of great significance in the development of matrix theory and related application fields.

## References

- [1] J. Liu, Y. Huang, Some properties on Schur complements of H-matrices and diagonally dominant matrices, Linear Algebra Appl. 389 (2004) 365-380.
- [2] J. Liu, J. Li, Z. Huang, X. Kong, Some properties of Schur complements and diagonal-Schur complements of diagonally dominant matrices, Linear Algebra Appl. 428 (2008) 1009-1030.
- [3] R. Smith, Some interlacing properties of the Schur complement of a Hermitian matrix, Linear Algebra Appl. 177 (1992) 137-144.
- [4] C. Li, Z. Huang, J. Zhao, On Schur complements of Dashnic-Zusmanovich type matrices, Linear Multilinear Algebra 70 (2022) 4071-4096.
- [5] J. Zhao, Q. Liu, C. Li, et al. Dashnic-Zusmanovich type matrices: a new subclass of nonsingular H-matrices [J]. Linear Algebra and its Applications, 2018, 552: 277-287.
- [6] C. Li, L. Cvetkovic, Y. Wei, et al. An infinity norm bound for the inverse of Dashnic-Zusmanovich type matrices with applications [J]. Linear Algebra and its Applications, 2019, 565: 99-122.
- [7] W. Zeng, J. Liu, H. Mo, Schur complement-based infinity norm bound for the inverse of Dashnic-Zusmanovich type matrices [J]. Mathematics, 2023, 11(10): 2254.
- [8] P. Dai, D. Pan. Subdirect sum of Dashnic-Zusmanovich type matrices [J]. Chinese journal of engineering mathematics, 2022, 39(06): 979-996.
- [9] L. Liu, X. Chen, Y. Li, et al. Subdirect sums of Dashnic-Zusmanovich matrices [J]. Bulletin des Sciences Mathématiques, 2021, 173: 103057.