

Numerical Study on the Synchronized Behavior and Amplitude-Frequency Characteristics of the Double Pendulum System under Certain Parameters

Zhentaο Liu

South China University of Technology, Guangzhou, China
202230352274@mail.scut.edu.cn

Abstract: This paper investigates the synchronized behavior and amplitude-frequency characteristics of the double pendulum system in conservative and non-conservative systems using numerical methods. A dynamical model is established based on Lagrangian mechanics, and the equations of motion are solved using the fourth-order Runge-Kutta method. By combining Monte Carlo sampling, bifurcation analysis, and phase correlation quantification methods, the influence mechanism of initial conditions and parameter adjustments on the system's steady-state evolution is revealed. The study demonstrates the following: (1) In the conservative system, the system's evolution highly depends on initial conditions, with synchronization and anti-synchronization states distributed in specific regions; (2) In the non-conservative system, external energy input and dissipation can reduce the system's dependence on initial conditions. Increasing energy loss suppresses chaos, and the driving frequency ω_F induces bifurcation under certain energy parameters, with the system exhibiting a dual-attractor coexistence phenomenon. The steady-state selection shows path dependence; (3) Parameter adjustments can significantly enhance the system's stability after symmetry breaking. In certain double pendulum mass and length ratios, the universality of the attractor-dominated synchronization mechanism in asymmetric systems is verified. This research provides a theoretical basis for the chaotic synchronization control of multi-joint mechanical systems and complex oscillator networks, revealing the key roles of energy dissipation and parameter optimization in suppressing chaos and achieving controllable synchronization.

Keywords: Double Pendulum System, Chaotic Synchronization, Amplitude-Frequency Characteristics, Bifurcation Analysis, Attractor, Numerical Simulation

1. Introduction

Chaos phenomena occupy an important position in the study of nonlinear physics and have universal significance in natural science fields across various scales, such as the butterfly effect in the atmosphere, synchronized firing of neurons, and the system stability phase transitions caused by small parameter perturbations in control engineering.

In complex systems, the coupling interactions and nonlinear characteristics between individual components give rise to a rich variety of dynamic behaviors. These behaviors often manifest as the coexistence or mutual switching of orderliness and randomness within the system. Research into the

dynamic behavior of nonlinear systems is instrumental in understanding these characteristics of complex systems. From the automated detection methods in biomedicine for noise resistance, which effectively address the reliable identification of chaotic phenomena in natural systems [1], to the theory of Conditional Nonlinear Optimal Perturbation (CNOP) used in analyzing parameter sensitivity in complex ecosystems, these approaches provide theoretical foundations for the nonlinear dynamics within complex systems [2].

In the study of classical chaotic systems, Lorenz first discovered chaotic phenomena in meteorological models and introduced the Lorenz attractor, which revealed the exponential sensitivity of the system to small initial perturbations [3]. Yoshiki Kuramoto proposed the mathematical model for phase oscillator synchronization, known as the Kuramoto model [4]. Bi et al. constructed a third-order dynamical model under the 1:1 resonance condition of the double pendulum system and revealed for the first time the evolutionary mechanism of Hopf bifurcation and chaos at the critical parameter boundary [5]. Stachowiak et al. analyzed the transition of the double pendulum system from low-energy regular motion to high-energy chaotic behavior using methods such as Lyapunov exponents through numerical computation [6]. Guptad et al. used numerical simulations to reveal that the chaotic behavior of the double pendulum system strengthens with increasing mass and length, providing new theoretical foundations for the parameter optimization and chaos control of the double pendulum dynamical system [7]. Wu et al. were the first to achieve stable anti-phase synchronization by controlling rolling friction in a coupled metronome model, revealing the dependence of synchronization on energy dissipation in systems with differing parameters [8].

The synchronization of coupled oscillator systems is a core issue in the research of nonlinear dynamics and complex systems science. Among these, the double pendulum model occupies a special place due to its simple mechanical structure and rich dynamical behavior. In this system, the sensitivity to initial conditions can cause the system's motion state to diverge exponentially; regular periodic motion can abruptly transition into a turbulent, disordered state when parameters change. This deterministic random behavior on a macroscopic scale makes the double pendulum system an important link between chaos theory research and industrial applications, such as multi-joint robot control and spacecraft attitude dynamics, in cutting-edge fields of study.

2. Model

This paper investigates the dynamic characteristics of the mathematical double pendulum system in both ideal isolated and forced states. A schematic diagram of the system is shown in Fig. 1.

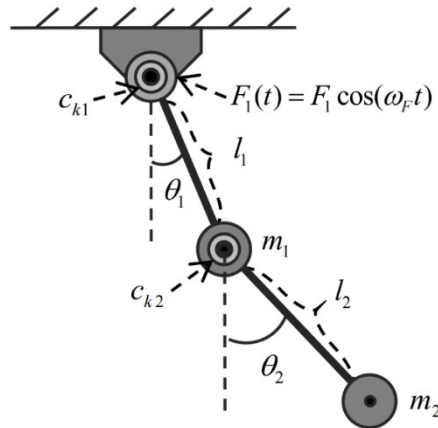


Figure 1: Schematic diagram of the double pendulum model

Based on theoretical mechanics analysis methods, the system's Lagrangian is given by Eq. (1) and Eq. (2). To explore the influence mechanism of initial conditions and system parameters on synchronization and steady-state evolution, the following double pendulum dynamic model is established:

$$T = \frac{1}{2}m_1\dot{l}_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2] \quad (1)$$

$$V = m_1gl_1(1 - \cos\theta_1) + m_2g[l_1(1 - \cos\theta_1) + l_2(1 - \cos\theta_2)] \quad (2)$$

The Lagrange equation is as follows:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_i}\right) - \frac{\partial T}{\partial \theta_i} = -\frac{\partial V}{\partial \theta_i} - c_{ki}\dot{\theta}_i + F_i(t) \quad (i = 1,2) \quad (3)$$

The resulting dynamic equations are:

$$(m_1 + m_2)l_1\ddot{\theta}_1 - (m_1 + m_2)g\sin\theta_1 + m_2l_2\sin(\theta_1 - \theta_2)\ddot{\theta}_2 + l_2\cos(\theta_1 - \theta_2)\dot{\theta}_2m_2 = -c_{k1}\dot{\theta}_1 + F_1\sin(\omega_{F1}t) \quad (4)$$

$$m_2l_2\ddot{\theta}_2 - m_2g\sin\theta_2 - m_2l_1\sin(\theta_1 - \theta_2)\ddot{\theta}_1 + m_2l_1\cos(\theta_1 - \theta_2)\dot{\theta}_1 = -c_{k2}\dot{\theta}_2 \quad (5)$$

Where θ_1, θ_2 are the generalized coordinates of the two pendulum bobs, c_{k1}, c_{k2} are the angular damping coefficients, l_1, l_2 are the lengths of the pendulums, m_1, m_2 are the masses of the pendulum bobs, and are the system frequency parameters. F_1 and ω_{F1} are the amplitude and frequency of the external driving force acting on θ_1 , and t is the time parameter.

3. Numerical results

In our numerical simulations, equations (4) and (5) are solved using the fourth-order Runge-Kutta method, with a time step of $dt=0.01$.

The system is within a four-dimensional phase space framework. In this experiment, the initial angular velocities of the two pendulums, $\dot{\theta}_i(0) = 0, i = 1,2$, are constrained, forming a zero-velocity hyperplane in the four-dimensional phase space at the initial state: $\sum_{t=0} = \{(\theta_1, \theta_2, 0,0) | \theta_1, \theta_2 \in \mathbb{R}\}$. That is, the experiment only discusses the effects on the four-dimensional phase space motion when the initial angles $\theta_i(0), i = 1,2$, are selected in the two-dimensional phase space at $t=0$.

3.1. Symmetric pendulum in a conservative system

In the experiment, symmetric pendulums are set as follows: $m_1 = 1, l_1 = 1, m_2 = m_1, l_2 = l_1, g = 9.80, dt = 0.01$. In the conservative system, $F_1 = 0, c_{k1} = 0, c_{k2} = c_{k1}$. The system is discussed under the Hamiltonian conservative system framework.

To study the effect of the initial potential energy on the system's final state evolution, the experiment generates random initial conditions using the Monte Carlo method within the hyperplane $\sum_{t=0} = \{(\theta_1, \theta_2, 0,0) | \theta_1, \theta_2 \in [-\pi, \pi]\}$. It was found that the system evolves into three typical states depending on the initial values, as shown in Fig. 2.

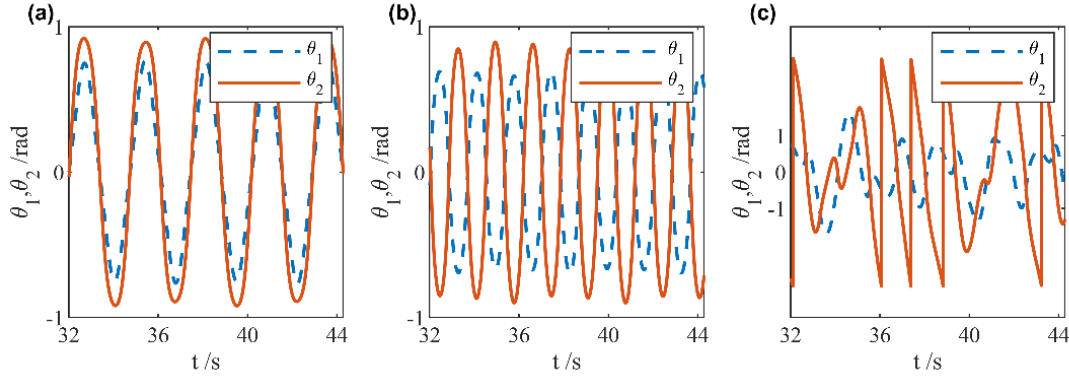


Figure 2: Time series evolution of the two pendulums (a) complete synchronization; (b) complete anti-synchronization; (c) chaos

To quantify the synchronization and anti-synchronization states of the system, the time series of the two pendulum angles at steady states are converted into phase diagrams, and linear correlation analysis is performed to obtain the phase time series linear correlation coefficient ρ . With initial conditions $\dot{\theta}_1 = 0, \dot{\theta}_2 = 0$ in the two-dimensional phase space, as shown in Fig. 2 (a), $\theta_1(0) = 0.7461, \theta_2(0) = 0.9233$, the system can reach a state of complete synchronization, where the linear correlation coefficient ρ between $\theta_1(t)$ and $\theta_2(t)$ is very close to 1. As shown in Fig. 2 (b), with initial conditions $\theta_1(0) = -0.6603, \theta_2(0) = 0.9005$, ρ is very close to -1, indicating that the system reaches a state of anti-synchronization. At larger initial angles, as shown in Fig. 2 (c), when $\theta_1(0) = 2.2268, \theta_2(0) = -1.0720$, the system enters a chaotic state.

To further explore the influence of different initial conditions on the steady state of the system in the hyperplane $\sum_{t=0} = \{(\theta_1, \theta_2, 0, 0) | \theta_1, \theta_2 \in [-\pi, \pi]\}$, the experiment generates 1100×1100 initial points by uniformly sampling within the two-dimensional phase space of $\theta_1(0), \theta_2(0) \in [-\pi, \pi]$. The results corresponding to the phase time series linear correlation coefficient ρ are shown in Fig. 3.

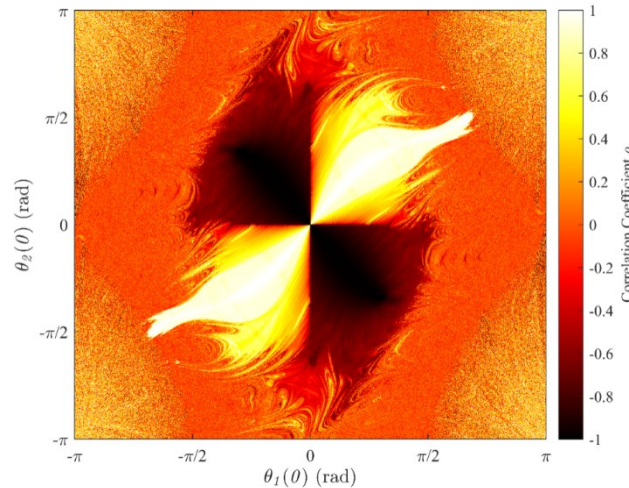


Figure 3: Distribution of the phase time series linear correlation coefficient ρ for the two pendulums in the initial phase two-dimensional phase space. The correlation coefficient ρ ranges from 1 (white) to -1 (black) sample size $n=1100 \times 1100$, phase space $\theta_i(0) \in [-\pi, \pi], i = 1, 2$

In Fig. 3, $\rho=1$ indicates that the two pendulums evolve to complete synchronization; $\rho=-1$ indicates that the two pendulums reach complete anti-synchronization. Samples with $\rho \in [-0.2, 0.2]$ account for 47.30%.

3.2. Symmetry-broken pendulum under non-conservative forces

In a conservative system under the Hamiltonian framework, the final steady-state exhibits a strong dependence on the initial conditions when the system's initial state lies on the hyperplane $\Sigma_{t=0} = \{(\theta_1, \theta_2, 0, 0) | \theta_1, \theta_2 \in [-\pi, \pi]\}$. In the case of forced vibrations, is it possible to disrupt the steady-state energy space defined by the Hamiltonian H through external energy input, thereby controlling the system's final evolutionary state?

To address this question, the experiment modifies the energy of the conservative system to establish a forced vibration system, investigating the changes in the system's steady state with the driving frequency as the primary influencing factor. Specifically, a driving force $F_1(t)$ is introduced to θ_1 , and equal oscillatory damping coefficients $c_{k2} = c_{k1}$ are applied to both θ_2 and θ_1 . Initial conditions use the Monte Carlo method within randomly selected within the two-dimensional phase space $\theta_1(0), \theta_2(0) \in [-\pi, \pi]$, and the driving frequency is scanned over the interval $\omega_F \in [0, 8]$.

The experiment takes the occurrence of single-period or multi-period phenomena under specific external energy parameters as an indicator of the system's evolution to a steady state. A bifurcation diagram is plotted for the maximum amplitude A of θ_2 at steady state as a function of the driving frequency ω_F , as shown in Fig. 4.

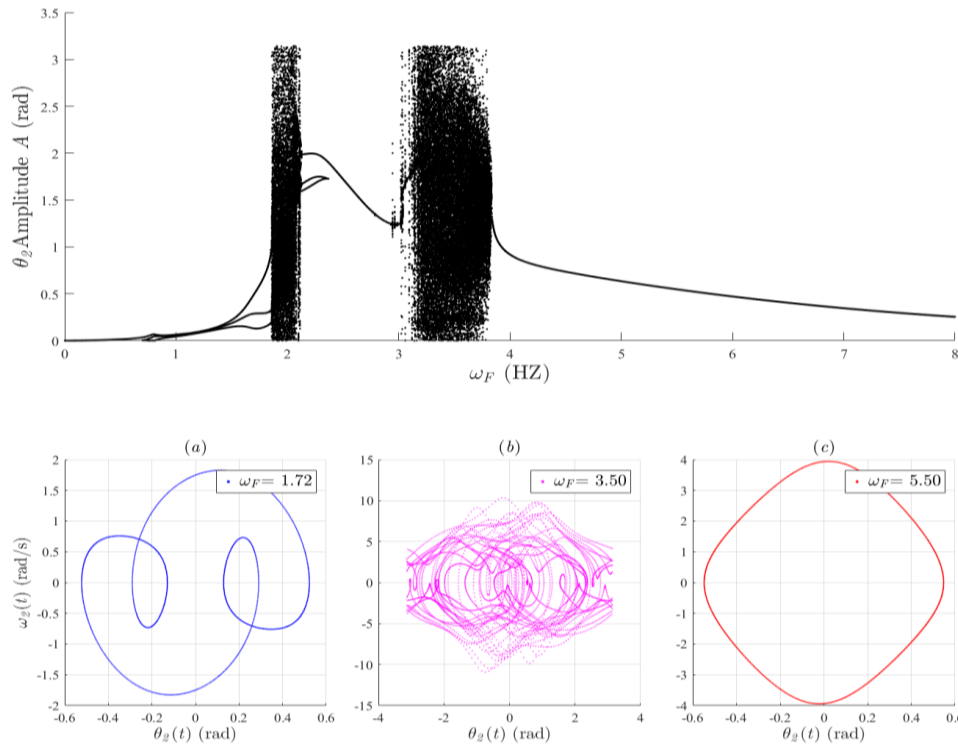


Figure 4: Presents the bifurcation diagram of θ_2 amplitude A under steady-state conditions, with ω_F uniformly sampled over the interval $[0, 8]$ from randomly selected initial conditions. The parameters used are $\dot{\theta}_1(0) = 0, \forall \theta_1(0) \in [-\pi, \pi], i = 1, 2, F_1 = 10, c_{k1} = 0.8, c_{k2} = c_{k1}$

The experiment identifies three typical regions in the bifurcation diagram (Fig. 4): a multi-period region, a chaotic region, and a single-period region. By plotting the phase diagrams of $\theta_2 - \theta_1$,

different evolutionary behaviors are observed. As shown in Fig. 4 (a), at $\omega_F = 1.72$, the system evolves into a multi-period bifurcation state; at $\omega_F = 3.50$, as shown in Fig. 4 (b), the system transitions into chaos; and at $\omega_F = 5.50$, as shown in Fig. 4 (c), the system settles into a single-period state.

According to the experimental findings, when external energy input is introduced, the single-period region indicates that the system converges to the same steady state in four-dimensional phase space regardless of the initial conditions. However, in the chaotic region, the system's evolutionary trajectory becomes ambiguous. To further explore whether system evolution in the frequency range near the chaotic region depends on initial conditions, the experiment adjusts the energy input parameters, considering the driving force frequency as the primary factor influencing steady-state behavior. With external energy parameters set to $F_1 = 4.5025$, $c_{k1} = 0.7$, $c_{k2} = c_{k1}$, a bifurcation diagram dominated by single-period behavior is obtained, as shown in Fig. 5.

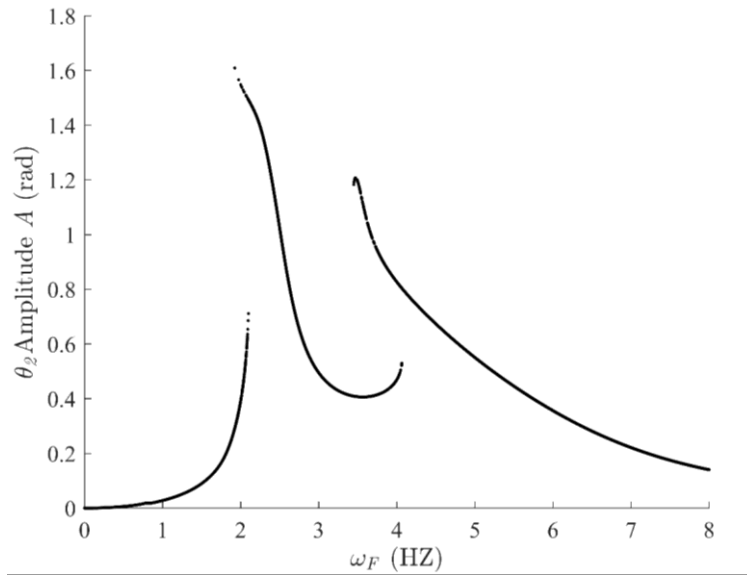


Figure 5: Presents the bifurcation diagram of θ_2 amplitude A under steady-state conditions, with ω_{F1} uniformly sampled over the interval $[0,8]$ from randomly selected initial conditions. The initial angular velocity is fixed at $\dot{\theta}_i(0) = 0, i = 1,2$, and initial angles $\forall \theta_i(0) \in [-\pi, \pi], i = 1,2$

From Fig. 5, it is evident that under external energy parameters $F_1 = 4.5025$, $c_{k1} = 0.7$, $c_{k2} = c_{k1}$, no chaotic regions appear within the frequency range $\omega_F \in [0,8]$. The experiment demonstrates that increasing damping appropriately suppresses symmetry breaking induced by the driving force. This suggests that the impact of symmetry breaking on system evolution has a certain elastic range. In the bifurcation diagram of amplitude A versus driving frequency, three frequency intervals exhibit strong continuity and partial overlap. Subsequent experiments focus on the bifurcation phenomena occurring in the high-frequency range $\omega_F \in [3.45, 3.93]$.

To determine whether the bifurcation within $\omega_F \in [3.45, 3.93]$ arises due to the system being attracted to two different attractors or whether it enters a multi-periodic state, the experiment performs a stability analysis of the bifurcation line. Once the system reaches a stable state, the driving force frequency is monotonically adjusted, allowing the system to evolve to a new steady state before repeating the process for the next frequency adjustment, as illustrated in Fig. 6.

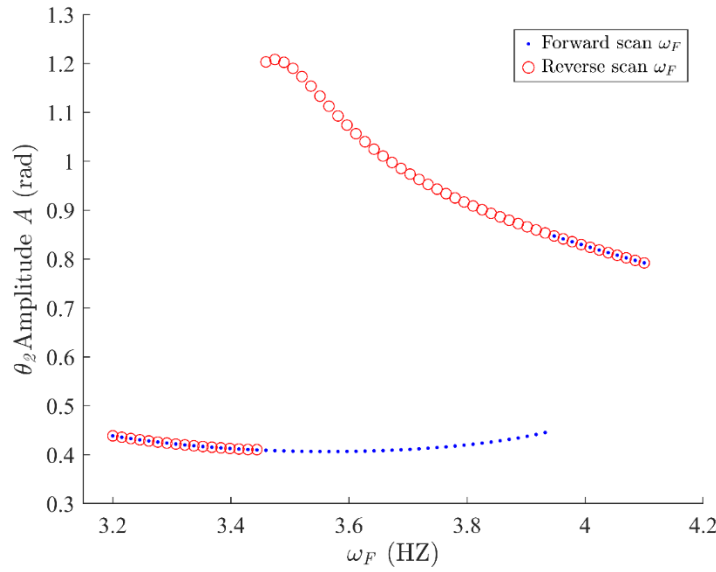


Figure 6: Presents the stability diagram for randomly selected initial system conditions, with sampling points uniformly taken from the frequency range $\omega_F \in [3.2, 4.1]$, where bifurcations exist in Fig. 5. The system's steady-state amplitudes A , which depend on both the magnitude and direction of the driving frequency θ_2 , are obtained through forward (blue dots) and backward (red circles) frequency sweeps

The experiment explores the system's steady states by adjusting the driving frequency $\omega_F \in [3.2, 4.1]$ through forward and backward sweeps, revealing two distinct cases in Fig. 6. The red circles correspond to decreasing the driving frequency while the system remains in a steady state, showing that as ω_F decreases, the amplitude θ_2 increases continuously until the red circles become discontinuous and transition to the regions marked by blue dots. Conversely, the blue dots correspond to increasing the driving frequency while the system remains in a steady state, demonstrating that as ω_F increases, the amplitude θ_2 initially decreases before increasing again, with transitions occurring at the red circle locations. This indicates that the system evolves into two different attractor basins in phase space. Overall, the experiment confirms that introducing external energy input effectively suppresses the influence of initial conditions on the system's steady state, ensuring that under specific external energy parameters, the steady-state motion of the system becomes independent of the initial conditions.

To further investigate the dependency of the system's steady state on initial conditions under the external energy parameters $F_1 = 4.5025$, $c_{k1} = 0.7$, $c_{k2} = c_{k1}$, the experiment analyzes the linear correlation of the phase time series for 20,000 randomly selected sample points $\theta_1(t)$, $\theta_2(t)$ within the frequency range $\omega_F \in [3.45, 3.93]$. After the introduction of external energy, the initial phase correlation coefficient near $\theta_1(0) \approx \theta_2(0)$ in the two-dimensional phase space exhibits inverse synchronization, which contrasts with the positive correlation observed in the conservative Hamiltonian system. Additionally, under these energy parameters, the overall system evolution predominantly trends toward inverse synchronization, as illustrated in Fig. 7.

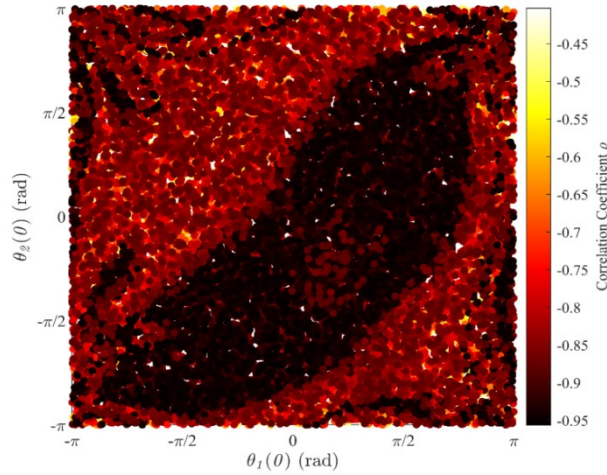


Figure 7: Presents the linear correlation coefficient ρ for the time series of 20,000 sample points within the multistable frequency range $\omega_F \in [3.45, 3.93]$. The correlation coefficient ρ ranges from 1 (white) to -1 (black). The initial two-dimensional phase space satisfies $\theta_i(0) \in [-\pi, \pi]$, $i = 1, 2$. Energy parameters: $F_1 = 4.5025$, $c_{k1} = 0.7$, $c_{k2} = c_{k1}$

The experiment demonstrates that under the parameter conditions $F_1 = 4.5025$, $c_{k1} = 0.7$, $c_{k2} = c_{k1}$, $\omega_F \in [3.45, 3.93]$, the system exhibits an attractor in phase space that leads to inverse synchronization, with no occurrence of in-phase synchronization between the two oscillators.

To further explore the distribution of attractors in phase space within the frequency range $\omega_F \in [3.45, 3.93]$, an analysis of attractor distribution is conducted within the two-dimensional initial phase space $\Sigma_{t=0} = \{(\theta_1, \theta_2, 0, 0) | \theta_1, \theta_2 \in [-\pi, \pi]\}$. Based on the bifurcation diagram in Fig. 5, the initial conditions for which θ_2 amplitude $A > 0.5$ are marked as red points, while those with $A < 0.5$ are marked as blue points, as shown in Fig. 8.

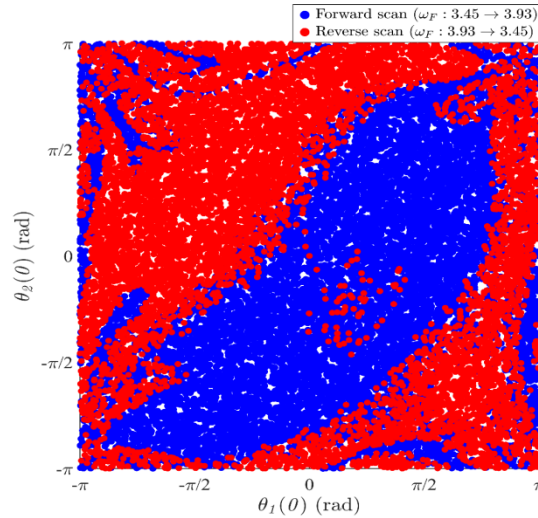


Figure 8: Presents the two-dimensional phase diagram of initial conditions within the multistable frequency range $\omega_F \in [3.45, 3.93]$. The sample size is 20,000, and the initial two-dimensional phase space satisfies $\forall \theta_i(0) \in [-\pi, \pi]$, $i = 1, 2$. Energy parameters: $F_1 = 4.5025$, $c_{k1} = 0.7$, $c_{k2} = c_{k1}$. The red points represent initial phases leading to stable states with $A > 0.5$, while blue points correspond to stable states with $A < 0.5$

The experiment found that the region of reverse synchronization in Fig. 7 overlaps significantly with the attraction basin marked by the blue points in Fig. 8; the attraction basin marked by the red points corresponds to the chaotic state in Fig. 7. Furthermore, within the region marked by the blue points, a chaotic region, resembling “islands,” appears where red points accumulate, and these “islands” are located in the fourth quadrant of the phase space, with $\forall \theta_i(0) \in [-\pi/2, -\pi/2], i = 1, 2$.

To further discuss the generality of the above results, an experiment was designed to break the symmetry of the system. The mass ratio and length ratio parameters were adjusted, keeping $m_2 = 1$ and $l_1 = 1$ fixed, while $m_1 = 0.5, l_2 = 2$ were varied. This means that the pendulum length of θ_1 , l_1 , was fixed, while the load m_1 was reduced; the load m_2 of θ_2 was fixed, and the pendulum length l_2 was increased. In this symmetry-broken system, the same external energy input was introduced to explore the attractor present in the system and its corresponding evolution relationship, with results consistent with the previous findings, as shown in Fig. 9.

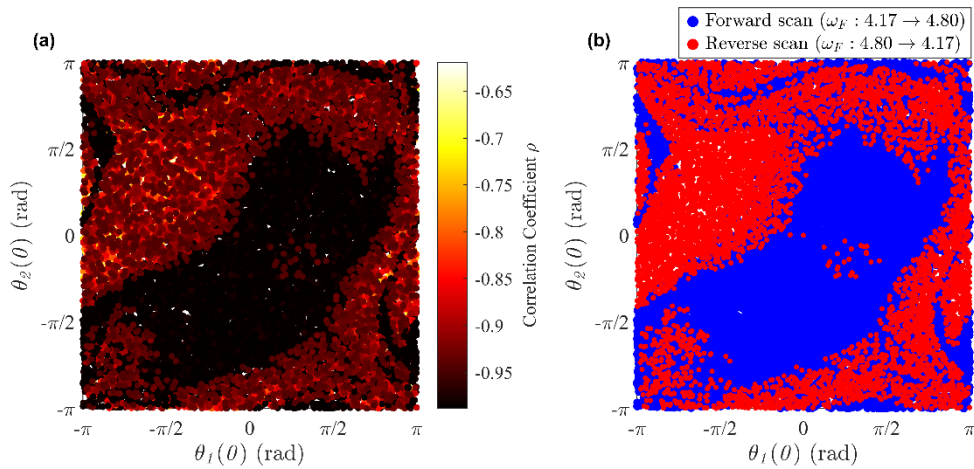


Figure 9: In the frequency range $\omega_F \in [4.17, 4.80]$, (a) the time-series linear correlation coefficient ρ of each sampling point is plotted. (b) The two-dimensional phase diagram of the initial conditions of the motion is shown. A total of 20,000 sample points were taken in phase space. $\forall \theta_i(0) \in [-\pi, \pi], i = 1, 2$. $m_1 = 0.5, m_2 = 1, l_1 = 1, l_2 = 2, c_{k1} = 0.6, F_1 = 4.5025, dt = 0.01, c_{k1} = c_{k2}$. (a) The initial phase points are randomly selected within $\theta_i(0) \in [-\pi, \pi]$, and the correlation coefficient ρ ranges from 1 (white) to -1 (black). (b) The blue represents stable states where $A < 0.2$ for the initial phase, while the red represents stable states where $A > 0.2$.

4. Conclusion

This study constructs a nonlinear dynamical model of a double pendulum system to reveal the universal principles governing chaotic synchronization and bifurcation behavior. It proposes a parameter-regulation-based strategy for chaos suppression and synchronization enhancement, providing a novel analytical framework for complex system control theory. The experiments demonstrate that the evolution of the double-pendulum oscillator in a Hamiltonian conservative system exhibits initial-condition dependence, with the system predominantly evolving into three states: synchronized, anti-synchronized, and chaotic. The synchronized state mainly occurs in the interval $\theta_1(0) \approx \theta_2(0)$ within the two-dimensional phase space $\Sigma_{t=0} = \{(\theta_1, \theta_2, 0, 0) \mid \theta_1, \theta_2 \in [-\pi, \pi]\}$, while the anti-synchronized state is primarily distributed in the interval $\theta_1(0) \approx -\theta_2(0)$. When both $\theta_1(0), \theta_2(0)$ have relatively large values, the system is more likely to evolve into a chaotic state.

For forced oscillators with external energy input, the system exhibits distinct dynamical characteristics dominated by the driving and damping forces. Additionally, symmetry breaking

caused by parameters such as mass ratio and pendulum length ratio also influences the system's dynamical behavior. This study primarily explores the impact of variations in the driving force frequency on the system's final state. For general energy parameters, the system's steady-state predominantly falls into three categories: single-period, period-doubling, and chaos. By appropriately reducing energy input and increasing energy dissipation, symmetry breaking in the system is partially suppressed, leading to a primarily single-state system. Moreover, in certain driving frequency ranges, the system's steady state becomes independent of its initial conditions. However, within specific frequency intervals, the system exhibits bistable attractor coexistence, where the attractors drive the system toward the anti-synchronized state. The effects of different attractors are directly reflected in the steady-state amplitude of the oscillators. When the mass ratio and pendulum length ratio are adjusted—altering the system's periodic parameters and inducing symmetry breaking—the system's final state remains consistent with the aforementioned states under specific energy input and dissipation conditions.

This study not only deepens the understanding of nonlinear behaviors in double-pendulum systems but also provides valuable insights for spacecraft attitude adjustment in microgravity and terrestrial robot control. The chaotic synchronization mechanism of the double-pendulum system can optimize the dynamic stability and disturbance resistance of multi-joint flexible robots, enhancing their adaptability in complex environments. Furthermore, through bifurcation analysis and anti-synchronization strategies, chaos suppression algorithms for multi-body spacecraft systems can be designed to reduce fuel consumption and the complexity of trajectory planning in deep-space exploration missions. These findings offer new perspectives for cutting-edge fields such as intelligent control and bionic engineering.

References

- [1] Toker, D., Sommer, F. T., & D'Esposito, M. (2020). *A simple method for detecting chaos in nature. Communications Biology*, 3, 11.
- [2] Ren, Q. J., Mu, M., Sun, G. D., & Wang, Q. (2023). *A new sensitivity analysis approach using conditional nonlinear optimal perturbations and its preliminary application. Advances in Atmospheric Sciences*, 40(2), 285–304. <https://doi.org/10.1007/s00376-022-1445-3>
- [3] Lorenz, E. N. (1963). *Deterministic nonperiodic flow. Journal of the Atmospheric Sciences*, 20(2), 130–141.
- [4] Kuramoto, Y. (1975). *International Symposium on Mathematical Problems in Theoretical Physics (H. Araki, Ed.), Lecture Notes in Physics*, 30, 420. Springer.
- [5] Bi, Q., & Chen, Y. (2000). *Bifurcation analysis of a double pendulum with internal resonance. Applied Mathematics and Mechanics*, 21, 255-264. <https://doi.org/10.1007/BF02459003>
- [6] Stachowiak, T., & Okada, T. (2006). *A numerical analysis of chaos in the double pendulum. Chaos, Solitons & Fractals*, 29, 417–422.
- [7] Gupta, M. K., Bansal, K., & Singh, A. K. (n.d.). *Mass and length dependent chaotic behavior of a double pendulum. Unpublished manuscript. University of Petroleum and Energy Studies and Visvesvaraya National Institute of Technology.*
- [8] Wu, Y., Wang, N., Li, L., & Xiao, J. (2012). *Anti-phase synchronization of two coupled mechanical metronomes. Chaos: An Interdisciplinary Journal of Nonlinear Science*, 22(2), 023146. <https://doi.org/10.1063/1.4729456>