

Closed Formula for the Number of Parts Great Spheres Divide an N-Sphere

Weijia Li

*Qingdao No.2 Middle School, Qingdao, China
vickyli318@163.com*

Abstract: This paper derives a recursive relationship and closed formula for the number of pieces of n dimensional sphere when divided by great spheres in general position. There are two main parts of this paper. The first is the proof of the recursive relationship and closed formula of a possibly known problem: the number of pieces separated by n copies of $(k - 1)$ -dimensional planes in general position in a k -dimensional Euclidean space. The second part is inspired by the first part, deriving a recursive relationship and closed formula for a similar problem but in spherical Geometry: the number of pieces separated by n copies of $(k - 1)$ -dimensional great spheres in general position in a k -dimensional sphere. As we will explain, the sphere problem we solve does not reduce to the planes problems we discuss first.

Keywords: Combinatorics, Spherical Geometry, Higher Dimension Geometry, Euclidean Geometry.

1. Introduction

It is a possibly known problem to figure out a recursive formula between $A_{(k,n)}$, $A_{(k,n-1)}$, and $A_{(k-1,n-1)}$ and finding the closed form of $A_{(k,n)}$ using mathematical induction. We can extend this problem from Euclidean Geometry to Spherical Geometry, which is the main problem that we are going to address in our essay. We are going to derive the recursive relationship of $B_{(k,n)}$, $B_{(k,n-1)}$, $B_{(k-1,n-1)}$ in Spherical Geometry by similar ways of deriving the recursive relationship in Euclidean Geometry, which is the idea of adding one additional $(k - 1)$ -dimensional plane and see how many new pieces of the subdivision does it add to the original configuration. In addition, to find the closed formula for this problem in Spherical Geometry, we are going to find out how the formulas for $A_{(k,n)}$ in Euclidean Geometry that are related to the formulas for $B_{(k,n)}$ in Spherical Geometry.

Remark: Note that our problem is philosophically related to the questions about the possible number of pieces that n lines in **not** general position could subdivide a plane and its higher dimensional versions, see O. Ivanov [1-3]

2. Definition

2.1. Definition for terms in euclidean geometry

2.1.1. $A_{(k,n)}$

Define $A_{(k,n)}$ as the number of regions that n copies of k -dimensional planes in general position divide a $(k + 1)$ -dimensional Euclidean space.

2.1.2. General position for k -dimensional planes

Define *general position* of hyperplanes in Euclidean Geometry as following. When there are n copies of an k -dimensional planes in a $(k + 1)$ -dimensional Euclidean space, arbitrarily take one of the k -dimensional planes (use P to denote this plane). If n copies of the k -dimensional planes are in general position, then the intersection between all other k -dimensional planes and P are $(k - 1)$ -dimensional planes that are in general position inside P . This should be the case for all planes P from our n planes. By similar means, all intersections at lower dimension are required to be in general position. This can be reduced all the way down to 1-dimensional planes (lines) in a 2-dimensional plane, which have to be lines in general position. The words “Lines in general position” means that any pair of lines intersect at exactly one unique point of intersection and in particular none of them are parallel or coincide.

2.2. Definition for terms in spherical geometry

2.2.1. k -dimensional sphere

Define a k -dimensional sphere as the surface described by the following equation in $\mathbb{R}^{k+1}$:

$$\sum_{i=0}^{i=k} x_i^2 = R^2, \text{ where } R \text{ denotes the radius of the sphere.}$$

2.2.2. $B_{(k,n)}$

Define $B_{(k,n)}$ as the number of regions that n copies of k -dimensional great spheres in general position divide the $(k + 1)$ -dimensional sphere.

2.2.3. Great k -dimensional sphere

Define k -dimensional great sphere as a k -dimensional sphere that has the same center and radius as the $(k + 1)$ -dimensional sphere that it is contained in, but possibly located not in the equatorial hyperplane. So it is a rotated version of the equatorial sphere. The rotated equator is the regular equator with respect to a rotated North Pole – South Pole Axis. The North Pole on the sphere with radius R is the point $(0, 0, \dots, 0, R)$ and the South Pole is the point $(0, 0, \dots, 0, -R)$.

2.2.4. General position for k -dimensional sphere

We define the general position for great spheres in Spherical Geometry in a way which is similar to that in Euclidean Geometry. When there are n copies of an k -dimensional great spheres in a $(k + 1)$ -dimensional sphere, arbitrarily take one of the k -dimensional great spheres (use S to denote this great sphere). If the n copies of the k -dimensional great spheres are in general position, then the intersection between all other k -dimensional great spheres and S are $(k - 1)$ -dimensional great spheres in S that are also in general position. This should be true for all sphere S from our collection. By similar means, all intersections at lower dimension are required to be in general position. This can

be reduced all the way down to 1-dimensional great spheres in general position within a 2-dimensional sphere, which are great circles in general position in a 2-dimensional sphere. The words “Great circles are in general position inside a 2-dimensional sphere” mean that no pair of great circles coincide with each other, and any pair of great circles intersect at exactly two unique points of intersection.

2.3. Introduction

It is a possibly known problem to figure out a recursive formula between $A_{(k,n)}$, $A_{(k,n-1)}$, and $A_{(k-1,n-1)}$ and finding the closed form of $A_{(k,n)}$ using mathematical induction. We can extend this problem from Euclidean Geometry to Spherical Geometry, which is the main problem that we are going to address in our essay. We are going to derive the recursive relationship of $B_{(k,n)}$, $B_{(k,n-1)}$, $B_{(k-1,n-1)}$ in Spherical Geometry by similar ways of deriving the recursive relationship in Euclidean Geometry, which is the idea of adding one additional $(k-1)$ -dimensional plane and see how many new pieces of the subdivision does it add to the original configuration. In addition, to find the closed formula for this problem in Spherical Geometry, we are going to find out how the formulas for $A_{(k,n)}$ in Euclidean Geometry that are related to the formulas for $B_{(k,n)}$ in Spherical Geometry.

Remark: Note that our problem is philosophically related to the questions about the possible number of pieces that n lines in not general position could subdivide a plane and its higher dimensional versions, see O. Ivanov [3]

3. Proof of main theorem

3.1. Deriving the recursive formula for $A_{(k,n)}$

3.1.1. Formula for $A_{(0,n)}$

$A_{(0,n)}$ means the number of regions that are created by n copies of the 0-dimensional planes (i.e. points) in general position subdividing a 1-dimensional plane (i.e. in a line). By adding the n^{th} point, there will be one additional subdivision of the line. Thus,

$$A_{(0,n)} = A_{(0,n-1)} + 1$$

So,

$$\sum_{m=2}^{m=n} A_{(0,m)} - A_{(0,m-1)} = \sum_{m=2}^{m=n} 1 = n - 1$$

Thus,

$$A_{(0,n)} - A_{(0,1)} = n - 1$$

Substituting,

$$A_{(0,1)} = 2,$$

We have

$$A_{(0,n)} = n + 1$$

3.1.2. Formula for $A_{(1,n)}$

$A_{(1,n)}$ means the number of regions that are created by n copies of the 1-dimensional planes (i.e. lines) in general position subdividing a 2-dimensional plane. By adding the n^{th} line, there will be additional $(n - 1)$ points of intersections of this line and of the previous line. This gives n parts of this new line each of which divides one of the previous planar pieces into two. Thus, we get n new regions of the plane.

Thus,

$$A_{(1,n)} = A_{(1,n-1)} + n$$

So,

$$\sum_{m=2}^{m=n} A_{(1,m)} - A_{(1,m-1)} = \sum_{m=2}^{m=n} m = \frac{1}{2}(n^2 + n - 2)$$

Thus,

$$A_{(1,n)} - A_{(1,1)} = \frac{1}{2}(n^2 + n - 2)$$

Substituting,

$$A_{(1,1)} = 2,$$

We have

$$A_{(1,n)} = \frac{1}{2}(n^2 + n + 2)$$

3.1.3. Recursive formula for $A_{(2,n)}$

By adding the n^{th} plane, then the other $(n - 1)$ planes will give $(n - 1)$ lines inside this n^{th} plane, which divide this added plane into $A_{(1,n-1)}$ regions, which gives us $A_{(1,n-1)}$ additional spatial regions since each of the $A_{(1,n-1)}$ planar regions subdivide one of the spatial regions into two.

Thus,

$$A_{(2,n)} = A_{(2,n-1)} + A_{(1,n-1)}$$

So,

$$\begin{aligned} \sum_{m=2}^{m=n} A_{(2,m)} - A_{(2,m-1)} &= \sum_{m=2}^{m=n} A_{(1,m-1)} \\ &= \sum_{m=2}^{m=n} \frac{(m-1)^2 + (m-1) + 2}{2} \end{aligned}$$

Thus,

$$A_{(2,n)} - A_{(2,1)} = \frac{1}{6}(n^3 + 5n - 6), \text{ see Faulhaber [2].}$$

Substituting,

$$A_{(2,1)} = 2$$

We have

$$A_{(2,n)} = \frac{1}{6}(n^3 + 5n + 6)$$

3.1.4. Recursive formula for $A_{(k,n)}$

By adding the n^{th} k -dimensional plane, the other $(n - 1)$ copies of k -dimensional planes will give additional $A_{(k-1,n-1)}$ pieces inside this n^{th} k -dimensional plane, which divides this added k -dimensional plane into $A_{(k-1,n-1)}$ pieces each of which subdivides one of the previous chambers of the spatial subdivision into two, thus giving us $A_{(k-1,n-1)}$ additional chambers.

Thus,

$$A_{(k,n)} = A_{(k,n-1)} + A_{(k-1,n-1)}$$

From former arguments, we can deduce that each of the general formula of number of chambers divided concerns the sum of powers of natural numbers. Here we can use Faulhaber's formula [2] to help us find the sum of p^{th} powers of the first n positive integers.

3.1.5. Deriving the closed formula for $A_{(k,n)}$

Before proving the closed formula for $A_{(k,n)}$, there is a lemma that needs to be proved:

Lemma:

$$A_{(r+1,n+1)} = A_{(r+1,1)} + \sum_{m=1}^{m=n} A_{(r,m)}$$

This lemma can be proved by induction on n .

Base case:

$n = 1$.

$A_{(r+1,n+1)} = A_{(r+1,2)} = A_{(r+1,1)} + A_{(r,1)}$ which holds

Inductive step:

Assume this lemma hold for all $n \leq a$

Consider the case where $n = a + 1$

$$\begin{aligned} A_{(r+1,n+1)} &= A_{(r+1,a+2)} = A_{(r+1,a+1)} + A_{(r,a+1)} \\ &= A_{(r+1,1)} + \sum_{m=1}^{m=a} A_{(r,m)} + A_{(r,a+1)} \\ &= A_{(r+1,1)} + \sum_{m=1}^{m=a+1} A_{(r,m)} \end{aligned}$$

Our lemma holds when $n = a + 1$

Thus, our lemma holds for all natural numbers n .

Now going back to deriving the closed formula for $A_{(k,n)}$:

By observation,

$$A_{(1,n)} = \binom{n}{2} + \binom{n}{1} + \binom{n}{0}$$

By mathematical induction on k , we can prove that for all natural number k and n ,

$$A_{(k,n)} = \sum_{m=0}^{m=k+1} \binom{n}{m}$$

Note that $\binom{n}{m} = 0$ for $m > n$.

The detailed proof is as below:

Base case: $k = 1$.

$$A_{(1,n)} = \frac{1}{2}(n^2 + n + 2) = \binom{n}{2} + \binom{n}{1} + \binom{n}{0} \text{ Which holds.}$$

Inductive step:

Assume $A_{(k,n)} = \sum_{m=0}^{m=k+1} \binom{n}{m}$, for all natural number n and $k \leq r$

Consider the case for $k = r + 1$

By the former proved lemma and Pascal's Identity[4], we can simplify our expression for $A_{(r+1,n+1)}$:

$$\begin{aligned} A_{(r+1,n+1)} &= A_{(r+1,1)} + \sum_{m=1}^{m=n} A_{(r,m)} = A_{(r+1,1)} + \sum_{m=1}^{m=n} \sum_{i=0}^{i=r+1} \binom{m}{i} \\ &= A_{(r+1,1)} + \sum_{i=0}^{i=r+1} \sum_{m=1}^{m=n} \binom{m}{i} \\ &= A_{(r+1,1)} + \sum_{i=0}^{i=r+1} \sum_{m=1}^{m=n} \left(\binom{m+1}{i+1} - \binom{m}{i+1} \right) \\ &= A_{(r+1,1)} + \sum_{i=0}^{i=r+1} \left(\binom{n+1}{i+1} - \binom{1}{i+1} \right) \\ &= A_{(r+1,1)} + \left(\sum_{i=0}^{i=r+1} \binom{n+1}{i+1} \right) - 1 \end{aligned}$$

Substituting $A_{(r+1,1)} = 2$

we have:

$$\begin{aligned} A_{(r+1,n+1)} &= 2 + \left(\sum_{i=0}^{i=r+1} \binom{n+1}{i+1} \right) - 1 = 1 + \sum_{i=0}^{i=r+1} \binom{n+1}{i+1} = \binom{n+1}{0} + \sum_{i=0}^{i=r+1} \binom{n+1}{i+1} \\ &= \sum_{i=0}^{i=r+2} \binom{n+1}{i} \end{aligned}$$

Thus,

$$A_{(r+1,n+1)} = \sum_{i=0}^{i=r+2} \binom{n+1}{i}$$

Our inductive hypothesis holds for $k = r + 1$

Thus, for all natural number k and n , $A_{(k,n)} = \sum_{m=0}^{m=k+1} \binom{n}{m}$, Where $\binom{n}{m}=0$ for $m>n$.

In fact, when $k > n$,

$A_{(k,n)} = \sum_{m=0}^{m=k+1} \binom{n}{m} = \sum_{m=0}^{m=n} \binom{n}{m} = 2^n$, by Binomial Theorem [1]

3.2. Deriving the recursive formula for $B_{(k,n)}$

3.2.1. Formula for $B_{(0,n)}$

$B_{(0,n)}$ means the number of pieces that n copies of 0-dimensional great spheres in general position divide a 1-dimensional sphere (i.e. circle). Note that 0-dimensional sphere is a pair of points with coordinate $-R$ and R respectively. Thus, $B_{(0,n)}$ is equivalent to the number of pieces that n pairs of points in general position divide a 1-dimensional sphere. The circle is subdivided by $2n$ points, and thus there are $2n$ regions (arcs in this case).

3.2.2. Formula for $B_{(1,n)}$

$B_{(1,n)}$ means the number of pieces that n copies of 1-dimensional great spheres in general position divide a 2-dimensional sphere, which is equivalent to the number of pieces that n great circles in general position divide a sphere. To derive the general formula for $B_{(1,n)}$, we have to firstly construct a configuration of great circles in general position. We start with a non-general but imaginable case, where the n circles are meridians on the sphere (see figure 1, where $n = 3$). Then, we further rotate the great circles about the center of the sphere so that they can be in general position while they remain to be great circles. We can see that at the initial position, the great circles are subdivided into $2n$ arcs, and thus there are $2n$ regions.

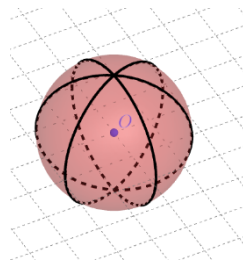


Figure 1: 3 great circles as meridians on a 2-dimensional sphere

Initially, we start with the case that any pair of great circles intersect at two same points, just as it is shown in figure 1. We arbitrarily pick a pair of great circles and let them remain unchanged. For the rest of the great circles, we rotate it around the center until it becomes general position, as exemplified in figure 2 where we take $n = 3$. Since the rotation is about the center, after rotation, the shape near the initial two points of intersection should be identical (to exemplify, see the two identical regions circled in figure 3) due to symmetry.

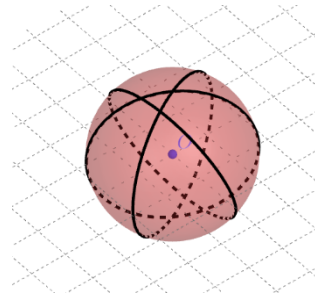


Figure 2: Rotated 3 great circles into general position on a 2-dimensional sphere

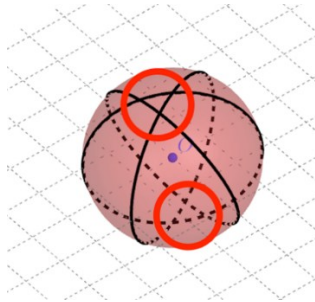


Figure 3: Additional regions created by rotation specified

We can reasonably imagine that the process of rotation is analogical to “expanding” the initial two points of intersection to two identical closed regions, which the areas circled out in red in figure 3 shows. The number of region in the closed piece divided by those great circles is equivalent to $A_{(1,n)} - 2n$, which is a trivial observation if we project the closed area onto a plane and see. Thus, after rotation from the initial position to a general position, there are $2(A_{(1,n)} - 2n)$ additional regions. Thus,

$$\begin{aligned} B_{(1,n)} &= 2n + 2(A_{(1,n)} - 2n) \\ &= 2A_{(1,n)} - 2n \\ &= n^2 + n + 2 - 2n \\ &= n^2 - n + 2 \end{aligned}$$

3.2.3. Recursive formula for $B_{(2,n)}$

$B_{(2,n)}$ means the number of pieces that n copies of 2-dimensional great spheres in general position divide a 3-dimensional sphere. When we add the n^{th} 2-dimensional great sphere S to $(n - 1)$ copies of the 2-dimensional great spheres in general position, there will be $(n - 1)$ additional 1-dimensional great spheres intersecting with the n^{th} 2-dimensional sphere S . By the definition of general position, those $(n - 1)$ copies of the 1-dimensional great spheres are also in general position within S . This gives $B_{(1,n-1)}$ additional pieces of the subdivision of S . Each additional piece of the subdivision of S subdivides one spatial region of the 3-dimensional sphere into two. Thus, $B_{(2,n)} = B_{(2,n-1)} + B_{(1,n-1)}$

$$\begin{aligned}
 \sum_{m=2}^{m=n} (B_{(2,m)} - B_{(2,m-1)}) &= \sum_{m=2}^{m=n} B_{(1,m-1)} = \sum_{m=2}^{m=n} (2A_{(1,m-1)} - 2(m-1)) \\
 &= \sum_{m=2}^{m=n} (2A_{(1,m-1)} - 2m + 2) \\
 &= 2 \left(\sum_{m=2}^{m=n} A_{(2,m)} - A_{(2,m-1)} \right) - 2 \left(\sum_{m=2}^{m=n} A_{(1,m)} - A_{(1,m-1)} \right) + 2(n-1) \\
 &= 2(A_{(2,n)} - A_{(2,1)}) - 2(A_{(1,n)} - A_{(1,1)}) + 2(n-1)
 \end{aligned}$$

Thus,

$$B_{(2,n)} - B_{(2,1)} = 2(A_{(2,n)} - A_{(2,1)}) - 2(A_{(1,n)} - A_{(1,1)}) + 2(n-1)$$

Substituting $A_{(2,1)} = 2, A_{(1,1)} = 2, B_{(2,1)} = 2$, we have

$$B_{(2,n)} = 2(A_{(2,n)} - A_{(1,n)}) + 2n$$

3.2.4. Recursive formula for $B_{(k,n)}$

$B_{(k,n)}$ means the number of pieces that n copies of k -dimensional great spheres in general position divide a $(k+1)$ -dimensional sphere. When we add the n^{th} k -dimensional great sphere S to $(n-1)$ copies of the k -dimensional great spheres in general position, there will be $(n-1)$ additional $(k-1)$ -dimensional great spheres intersecting with the n^{th} k -dimensional sphere S . By the definition of general position, those $(n-1)$ copies of the $(k-1)$ -dimensional great spheres are also in general position within S . This gives $B_{(k-1,n-1)}$ additional pieces of the subdivision of S . Each additional piece of the subdivision of S subdivides one spatial region of the $(k+1)$ -dimensional sphere into two. Thus, $B_{(k,n)} = B_{(k,n-1)} + B_{(k-1,n-1)}$

Remark: Note that the recursive formula for $B_{(k,n)}$ looks very similar to the recursive formula for $A_{(k,n)}$ however none of them can be directly obtained from each other. On the first glance the reduction of the planar case to the spherical case can be obtained by adding one point at infinity of the Euclidean space to get a sphere from it. This is the so called 1-point compactification of the Euclidean space. However, the compactified k -dimensional planes would not be round spheres and definitely would not be in general position since they all intersect at the added infinity point. The reduction in the other direction is not possible for similar reasons.

3.2.5. Closed formula for $B_{(k,n)}$

By observation, the closed formula for $B_{(k,n)}$ is:

$$\begin{cases} k \text{ is odd: } B_{(k,n)} = 2 + 2 \sum_{m=0}^{m=k} (-1)^{m+1} A_{(m,n)} \\ k \text{ is even: } B_{(k,n)} = -2 + 2 \sum_{m=0}^{m=k} (-1)^m A_{(m,n)} \end{cases}$$

Where n and k are natural numbers.

We can simplify the above expressions by using the closed formula for $A_{(k,n)}$, which is $A_{(k,n)} = \sum_{m=0}^{m=k+1} (-1)^m \binom{n}{m}$ as proved in 3.1.5

When k is odd:

$$B_{(k,n)} = 2 + 2 \sum_{m=0}^{m=k} (-1)^{m+1} A_{(m,n)} = 2 + 2 \sum_{m=0}^{m=k} (-1)^{m+1} \sum_{i=0}^{i=m+1} \binom{n}{i}$$

Through double counting, we can consider how many times each individual binomial number has appeared, and we can derive that

$$\sum_{m=0}^{m=k} (-1)^{m+1} \sum_{i=0}^{i=m+1} \binom{n}{i} = -\binom{n}{0} + \sum_{m=0}^{m=\frac{k+1}{2}} \binom{n}{2m}$$

By substituting this result back into the original closed formula for $B_{(k,n)}$ when k is odd, we have

$$B_{(k,n)} = 2 - 2 \binom{n}{0} + 2 \sum_{m=0}^{m=\frac{k+1}{2}} \binom{n}{2m} = 2 \sum_{m=0}^{m=\frac{k+1}{2}} \binom{n}{2m}$$

When k is even:

$$B_{(k,n)} = -2 + 2 \sum_{m=0}^{m=k} (-1)^m A_{(m,n)} = -2 + 2 \sum_{m=0}^{m=k} (-1)^m \sum_{i=0}^{i=m+1} \binom{n}{i}$$

Through double counting, we can consider how many times each individual binomial number has appeared, and we can derive that

$$\sum_{m=0}^{m=k} (-1)^m \sum_{i=0}^{i=m+1} \binom{n}{i} = \binom{n}{0} + \sum_{m=0}^{m=\frac{k}{2}} \binom{n}{2m+1}$$

By substituting this result back into the original closed formula for $B_{(k,n)}$ when k is even, we have

$$B_{(k,n)} = -2 + 2 \binom{n}{0} + 2 \sum_{m=0}^{m=\frac{k}{2}} \binom{n}{2m+1} = 2 \sum_{m=0}^{m=\frac{k}{2}} \binom{n}{2m+1}$$

After this simplification, we can write out the simplified closed formula for $B_{(k,n)}$ and then prove it using induction.

The closed formula for $B_{(k,n)}$ after simplification is:

$$\begin{cases} k \text{ is odd: } B_{(k,n)} = 2 \sum_{m=0}^{m=\frac{k+1}{2}} \binom{n}{2m} \\ k \text{ is even: } B_{(k,n)} = 2 \sum_{m=0}^{m=\frac{k}{2}} \binom{n}{2m+1} \end{cases}$$

This formula can be proved by mathematical induction on k .

Base case:

when $k = 0$, $B_{(0,n)} = 2n = 2 \binom{n}{0}$ which holds.

When $k = 1$, $B_{(1,n)} = 2A_{(1,n)} - 2n = 2\binom{n}{0} + 2\binom{n}{1} + 2\binom{n}{2} - 2n = 2\binom{n}{0} + 2\binom{n}{2}$, which holds.

Inductive step:

Assume the closed formula holds for all k when $k \leq r$. Consider the case where $k = r + 1$

Before the inductive step, we need to clarify an identity:

$$B_{(r+1,n+1)} = B_{(r+1,1)} + \sum_{m=1}^{m=n} B_{(r,m)} = 2 + \sum_{m=1}^{m=n} B_{(r,m)}$$

Since $B_{(k,n)}$ has exactly the same recursive formula with $A_{(k,n)}$, this identity can be derived using exactly the same method as we derived the same lemma for $A_{(r+1,n+1)}$, which was proved using induction in section 3.1.5

Substitute this back to our original equation, we would obtain $B_{(r+1,n+1)} = 2 \sum_{i=0}^{i=\frac{r+1}{2}} \binom{n+1}{2i+1}$

Case 1: If r is even, then $r + 1$ is odd.

$$\begin{aligned} B_{(r+1,n+1)} &= B_{(r+1,1)} + \sum_{m=1}^{m=n} B_{(r,m)} = 2 + \sum_{m=1}^{m=n} B_{(r,m)} = 2 + \sum_{m=1}^{m=n} 2 \sum_{i=0}^{i=\frac{r}{2}} \binom{m}{2i+1} \\ &= 2 + 2 \sum_{i=0}^{i=\frac{r}{2}} \sum_{m=1}^{m=n} \binom{m}{2i+1} = 2 + 2 \sum_{i=0}^{i=\frac{r}{2}} \left(-\binom{0}{2i+1} + \sum_{m=0}^{m=n} \binom{m}{2i+1} \right) \\ &= 2 + 2 \sum_{i=0}^{i=\frac{r}{2}} \sum_{m=0}^{m=n} \binom{m}{2i+1} \end{aligned}$$

We can simplify $\sum_{m=0}^{m=n} \binom{m}{2i+1}$ by Chu's Theorem [4-5]:

$$\sum_{m=0}^{m=n} \binom{m}{2i+1} = \binom{n+1}{2i+2}$$

Substitute this back to our original equation, we would obtain

$$\begin{aligned} B_{(r+1,n+1)} &= 2 + 2 \sum_{i=0}^{i=\frac{r}{2}} \binom{n+1}{2i+2} = 2 + 2 \sum_{m=1}^{m=\frac{r+2}{2}} \binom{n+1}{2m} \\ &= 2 \binom{n+1}{0} + 2 \sum_{m=1}^{m=\frac{r+2}{2}} \binom{n+1}{2m} = 2 \sum_{m=0}^{m=\frac{r+2}{2}} \binom{n+1}{2m} \end{aligned}$$

Case 2: If r is odd, then $r + 1$ is even

$$\begin{aligned} B_{(r+1,n+1)} &= B_{(r+1,1)} + \sum_{m=1}^{m=n} B_{(r,m)} = 2 + \sum_{m=1}^{m=n} B_{(r,m)} = 2 + \sum_{m=1}^{m=n} \left(2 \sum_{i=0}^{i=\frac{r+1}{2}} \binom{m}{2i} \right) \\ &= 2 + 2 \sum_{i=0}^{i=\frac{r+1}{2}} \sum_{m=1}^{m=n} \binom{m}{2i} = 2 + 2 \sum_{i=0}^{i=\frac{r+1}{2}} \left(-\binom{0}{2i} + \sum_{m=0}^{m=n} \binom{m}{2i} \right) \\ &= 2 - 2 \sum_{i=0}^{i=\frac{r+1}{2}} \binom{0}{2i} + 2 \sum_{i=0}^{i=\frac{r+1}{2}} \sum_{m=0}^{m=n} \binom{m}{2i} \\ &= 2 - 2 \binom{0}{0} + 2 \sum_{i=0}^{i=\frac{r+1}{2}} \sum_{m=0}^{m=n} \binom{m}{2i} = 2 \sum_{i=0}^{i=\frac{r+1}{2}} \sum_{m=0}^{m=n} \binom{m}{2i} \end{aligned}$$

We can simplify $\sum_{m=0}^{m=n} \binom{m}{2i}$ by Chu's theorem [5]:

$$\sum_{m=0}^{m=n} \binom{m}{2i} = \binom{n+1}{2i+1}$$

Thus our inductive hypothesis is true for $k = r + 1$.

4. Conclusion

There are three kinds of 2-dimensional geometries: Euclidean, Spherical, and Hyperbolic. It seems that such formulas do not admit generalization to Hyperbolic Geometry. The natural generalization of planes and great spheres to Hyperbolic geometry are the so called totally geodesic hyperbolic submanifolds, but they do not have to intersect at all. Thus, in Hyperbolic Geometry, it is likely not possible that we can derive a general recursive formula for the number of pieces that such submanifolds in general position subdivide the Hyperbolic space, as they do not have to intersect.

Acknowledgement

I would like to convey my greatest gratitude towards those who had always been supporting me. Firstly, I am deeply grateful to Professor Vladimir Chernov for posing this fascinating mathematical problem and offering invaluable guidance, insights and encouragement. Your expertise and patience was vital in shaping this research, and I am truly fortunate to work under your mentorship. In addition, I would also like to extend my sincere thanks to Mr. Evan Huang for giving such constructive help and support along the way. Your thoughtful advices and willingness to help had made a real difference to this paper, and I genuinely appreciate it. Moreover, I would like to thank Neoscholar Company for providing me with this opportunity to conduct the great research activity. Lastly, I would like to thank my family and friends: thank you so much for always loving me and encouraging me to pursue my passion in mathematics, your love and support truly meant a lot to me.

References

- [1] Brualdi, Richard A. *Introductory Combinatorics*. Pearson/Prentice Hall, 2010. p. 130. ISBN 978-0-13-602040-0.
- [2] Faulhaber, J. *Academia Algebrae, Darinnen die miraculosische Inventiones zu den höchsten Cossen weiters continuirt und profitiert werden*. Augspurg [sic], Germany: Johann Ulrich Schöning, 1631.
- [3] O. Ivanov *On the Number of Regions into Which n Straight Lines Divide the Plane* *The American Mathematical Monthly* Vol. 117, No. 10 (December 2010), pp. 881-888 (8 pages) Published By: Taylor & Francis, Ltd.
- [4] Mazur, David R. *Combinatorics: A Guided Tour*. Mathematical Association of America, 2010. p. 60 ISBN 978-0-88385-762-5.
- [5] Merris, Russell (2003). *Combinatorics* (2nd ed.). Hoboken, N.J.: Wiley-Interscience. p. 45. ISBN 0-471-45849-X.