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Quantum Cost of Dense Coding of Noisy and Multiple Use of Bell Channels

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Abstract: Dense coding is a basic protocol for quantum information transfer. We study the noisy and multi-use of quantum channels for dense coding protocols. For this purpose, we replace the standard channel, namely the EPR pair by a noisy two-qubit pure state. We construct the measurement Alice carries out to encode her message to be transferred to Bob. It turns out that the resulting states are not all orthogonal, and we propose a probability function for success transfer. We mathematically maximize the function in terms of the quantum measurements taken by Bob. We further develop the protocol to multi-use channels, and investigate the probability function for success transfer.

Keywords: Dense coding, Quantum entanglement, Qubit.

1. Introduction

In the past decades, various quantum protocols such as teleportation and dense coding assisted by classical channels has been proposed [1,2]. Next, it has been shown that the operational principles and mathematical foundations of several existing quantum communication protocols Ref. [3,4]. Third, quantum protocols with entanglement works for classical algorithms and calculations [5]. The recent development on classical encryption methods are not entirely secure, partially due to that the methods for quickly breaking classical encryption have been proposed [6]. In recent years, the idea of addressing issues such as decoherence and noise [7,8] and the unique advantages of quantum encryption are have been well studied [9,10]. It has been shown that the practical steps for quantum information transmission protocols can be used to reduce costs and enhance efficiency [11]. Further, the quantum cost of dense coding and teleportation protocols has been evaluated in terms of CNOT gates [12]. It has been also shown that the quantum remote control may be applied to realize the teleportation unitary gates [13]. A similar skill was used to transfer angles with a secure way by quantum teleportation [14]. Further, multipartite scenarios have been considered for a deterministic single-qubit. These progress and development have substantially propelled the potential applications of quantum communication and networking forward [15]. Besides, operation sharing with five-qubit cluster state has been established [16,17]. These operations and protocols provide a good foundation for realizing more complex protocols of transmitting information securely.

In this paper we study the noisy and multi-use of quantum channels for dense coding protocols. Firstly we review the fundamental protocols of system A and B dense coding. Then we study the noisy quantum-channel-assisted dense coding protocol between A and B. For this purpose, we replace the standard channel, namely the EPR pair by a noisy two-qubit pure state. We propose the measurement Alice carries out to encode her message to be transferred to Bob. We show that the resulting states are not all orthogonal, and thus it is impossible to discriminate them with probability one. We propose a probability function for success transfer, which need be maximized in terms of the quantum measurements taken by Bob. It turns out that the probability function is related to the parameters in the noisy channel. Based on the above results, we develop the protocol to multi-use channels, and investigate the probability function for success transfer with more complex technologies. We show our result in Figure 2.

The rest of this paper is organized as follows. In Sec. 2 we introduce the preliminary knowledge used for this paper. In Sec. 3 we introduce the main result of this paper. Then we conclude in Sec. 4.

2. Preliminaries

We introduce the basic knowledge of this paper. In Sec. 2.1, we review the knowledge of complex numbers. In Sec. 2.2 we introduce the notion of Kronecker product. Then in Sec. 2.3 we review the basic knowledge on regular product of two matrices. Third in Sec. 2.4 we introduce the quantum states used in this paper.

2.1. Complex number and vector

In mathematics, the complex number is the most general number. It can be represented as $x = a + bi \in \mathbb{C}$ where $a, b \in \mathbb{R}$ and the imaginary unit $i = \sqrt{-1}$. The modulus of x is denoted as $|x| = (a^2 + b^2)^{1/2}$, and the complex conjugate of x is written as $x^* = a - bi$. Let $\mathbb{C}^n$ be the n-dimensional Hilbert space over the complex field C. The modulus of a vector $y = \begin{bmatrix} y_1 \\ \cdots \\ y_n \end{bmatrix} \in \mathbb{C}^n$ can also be denoted as $|y|$, which equals to $\left(\sum_{j=1}^{n} |y_j|^2\right)^{1/2}$. Further we denote the complex conjugate transpose of y as $y^\dagger = [y_1^* \quad \ldots \quad y_n^*]$. Let another vector $z = \begin{bmatrix} z_1 \\ \cdots \\ z_n \end{bmatrix} \in \mathbb{C}^n$. In this case, the inner product of y and z equals to $< y, z >: = y^\dagger z = \sum_{j=1}^{n} y_j^* \, z_j$. In particular, if the inner product is zero then we say that y and z are orthogonal vectors. If $y = z$, then the (standard) norm of vector y is defined as the inner product of y and y up to a square root, i.e.,

$$\parallel y \parallel: = (< y, y >)^{1/2} = \left(\sum_{j=1}^{n} |y_j|^2\right)^{1/2}. \tag{1}$$

In particular, if $\parallel y \parallel = 1$ then we say that y is a unit vector. In quantum physics, if we respectively denote y, z as $|y\rangle, |z\rangle$, then their inner product can be denoted as $\langle y|z\rangle = y^\dagger z$.

Using Euler's formula $e^{ic} = \cos c + i \sin c$ for any real c, we can derive that $x = a + bi$ also equals to $(a^2 + b^2)^{1/2} e^{i\theta} = |x| e^{i\theta}$ for some real θ. The derivation process is shown below.

$$\begin{aligned} x = {} & a + bi \\ = {} & (a^2 + b^2)^{1/2}\left(a/(a^2 + b^2)^{1/2} + ib/(a^2 + b^2)^{1/2}\right) \\ = {} & (a^2 + b^2)^{1/2}(\cos\theta + i\sin\theta) \\ = {} & (a^2 + b^2)^{1/2} e^{i\theta}. \end{aligned} \tag{2}$$

One can verify that $\left(f e^{i\theta}\right)^* = f e^{-i\theta}$ for any real f and θ. Further, $f e^{i\alpha} \cdot g e^{i\beta} = f g e^{i(\alpha+\beta)}$ for any real f, g, α, β.

2.2. Kronecker Product

Definition 1. *Let A be an $n \times p$ matrix and B an $m \times q$ matrix. Then, as shown in FIG. 1, the Kronecker product of A and B ($A \otimes B$) is an $mn \times pq$ matrix.* ⊔

$$\boldsymbol{A} \otimes \boldsymbol{B} = \begin{bmatrix} a_{1,1}\boldsymbol{B} & a_{1,2}\boldsymbol{B} & \cdots & a_{1,p}\boldsymbol{B} \\ a_{2,1}\boldsymbol{B} & a_{2,2}\boldsymbol{B} & \cdots & a_{2,p}\boldsymbol{B} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,1}\boldsymbol{B} & a_{n,2}\boldsymbol{B} & \cdots & a_{n,p}\boldsymbol{B} \end{bmatrix}$$

Figure 1: Kronecker Product

2.3. Regular Product

The regular product of two matrices only exists when the column number of the first matrix is equal to the row number of the second matrix. If A is an $n \times p$ matrix and B is an $p \times m$ matrix, then the product matrix AB is an $n \times m$ matrix. To find the (a, b) element of the product matrix, we need to look at the a'th row of A and the b'th column of B. Then the inner product of the preceding row and column vectors is the (a, b) element of AB.

2.4. Quantum State

A quantum bit (qubit) state is a 2-dimensional unit vector, which can be denoted as $|x\rangle \in \mathbb{C}^2$ without referring to the particular function used to represent it. In many cases, for convenience we shall define

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{3}$$

The general expression of a qubit is $|x\rangle = \begin{bmatrix} \cos\alpha \\ e^{i\beta}\sin\alpha \end{bmatrix}$ where $\alpha \in [0, \pi/2]$ and $\beta \in [0,2\pi)$. Note that we do not need the global phase because it has no physical meaning in quantum physics.

Physically, one can use qubits to describe the condition of different particles, such as photons, atoms and molecules and so on. For example, if $|a\rangle$ and $|b\rangle$ are two qubits, then we can write $|\alpha\rangle = |a\rangle \otimes |b\rangle := |a, b\rangle$ as the condition of system A and B. We say that $|\alpha\rangle$ is a product state which has no quantum correlation at all. The reason is that, if system A is measured by physical operations, then system B is not affected, and vice versa. The entangled state $|\beta\rangle$ is a non-product state, i.e., $|\beta\rangle \neq |a, b\rangle$ for any qubits $|a\rangle$ and $|b\rangle$.

For instance, the two-qubit EPR pair (i.e., Bell state) $(|0,0\rangle + |1,1\rangle)/\sqrt{2}$ can be proved to be an entangled state. The proof is shown below.

$$\begin{gathered} |0,0\rangle = |0\rangle \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\ |1,1\rangle = |1\rangle \otimes |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \\ (|0,0\rangle + |1,1\rangle)/\sqrt{2} = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) /\sqrt{2} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \end{gathered} \tag{4}$$

Suppose the two-qubit Bell state is not entangled. We may assume that

$$\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = |x, y\rangle = \begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac \\ ad \\ bc \\ bd \end{bmatrix} \tag{5}$$

So we have

$$\begin{gathered} ac = bd = 1/\sqrt{2} \\ a, b, c, d \neq 0 \\ ad, bc \neq 0 \end{gathered} \tag{6}$$

Thus, we can prove that

$$(|0,0\rangle + |1,1\rangle)/\sqrt{2} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \neq |x, y\rangle. \tag{7}$$

So we have shown that the two-qubit Bell state is an entangled state. In some references, the Bell state is denoted as $(|00\rangle + |11\rangle/\sqrt{2}$.

2.5. Identity Matrix

An identity matrix, sometimes called a unit matrix, denoted as I_n, is an $n \times n$ matrix with all its diagonal elements equal to 1, and zeroes everywhere else. The expression of an identity matrix is shown below.

$$\mathrm{diag}(1,1,\ldots,1) := \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 \end{bmatrix}. \tag{8}$$

Suppose A and B are two $n \times n$ matrices. If their product $AB = I_n$, then A is the inverse of B and vice versa. For example, the inverse of a diagonal matrix $diag(a_1, a_2, \cdots, a_n)$ with nonzero a_j's is $diag(1/a_1, 1/a_2, \cdots, 1/a_n)$. In particular, the inverse of an identity matrix is also

2.6. Unitary Matrix

An $n \times n$ matrix U is unitary when $U * U^{\dagger} = I_n$. In quantum mechanics, the probability of successfully applying an unitary matrix (i.e., operation or gate) is 100%. Hence, researchers prefer to use unitary operations in quantum protocols. For example, the two-qubit CNOT and single-qubit Hadamard gates are commonly used unitary gates. The expressions of a CNOT gate and a Hadamard gate are shown below.

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{9}$$

$$H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{10}$$

2.7. Dense coding protocol

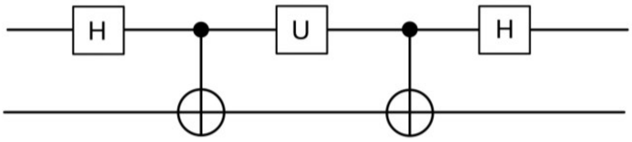

Figure 2: The CNOT and Hardmard gates H. The symbol U represents the identity matrix or one of the three Pauli gates σ_x, σ_y and σ_z.

In this subsection, we review the fundamental idea of dense coding protocol. The effect of applying CNOT gate to a two-qubit state $|a, b\rangle$ where $a, b = 0$ or 1 is shown below.

$$CNOT|a,b\rangle = |a, a \oplus b\rangle. \tag{11}$$

We shall perform one of three Pauli gates in the dense coding protocol, their expressions are given as $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

To perform a dense coding protocol, we firstly need a source S that generates an EPR pair shared by Alice and Bob. One example of an EPR state is shown below,

$$|\phi^+\rangle = (|0,0\rangle + |1,1\rangle)/\sqrt{2}. \tag{12}$$

Then, the source sends one member of the pair to Alice and the other to Bob. In classical communication, there are four values of two classical bits that Alice wish to send to Bob: 00, 01, 10 and 11. In quantum communication, each of these classical bits corresponds with one of these unitary operations: I, σ_x, σ_z, and $i\sigma_y$.

So if Alice wishes to send 00, she shall operate I, leading to

$$I \otimes I|\phi^+\rangle = |\phi^+\rangle. \tag{13}$$

If she wishes to send 01, she shall operate σ_x, leading to

$$\sigma_x \otimes I|\phi^+\rangle = |\psi^+\rangle = (|0,1\rangle + |1,0\rangle)/\sqrt{2}. \tag{14}$$

If she wishes to send 10, she shall operate σ_z, leading to

$$\sigma_z \otimes I|\phi^+\rangle = |\phi^-\rangle = (|0,0\rangle - |1,1\rangle)/\sqrt{2}. \tag{15}$$

If she wishes to send 11, she shall operate $i\sigma_y$, leading to

$$i\sigma_y \otimes I|\phi^+\rangle = |\psi^-\rangle = (|0,1\rangle - |1,0\rangle)/\sqrt{2}. \tag{16}$$

Subsequently, Alice shall sends her half of the EPR pair to Bob. Bob then needs to perform the unitary operations B on the EPR pair to convert the two qubits to the two classical bits.

$$B = (H \otimes I)CNOT = 1/\sqrt{2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}, \tag{17}$$

It is easy to get that

$$\begin{aligned} B|\phi^+\rangle &= |0,0\rangle, \quad B|\psi^+\rangle = |0,1\rangle, \\ B|\phi^-\rangle &= |1,0\rangle, \quad B|\psi^-\rangle = |1,1\rangle. \end{aligned} \tag{18}$$

At last, Bob measures the two qubits in the computational basis, acquiring with unit probability the two desired classical bits.

We still need to investigate the security of dense coding protocol by calculating the reduced density operators of particle A and B. For example, one can obtain the reduced density operator of particle B of $|\psi^+\rangle$ as follows.

$$\rho_B = \sum_{j=0}^{1}(\langle j| \otimes I_2)|\psi^+\rangle\langle\psi^+|(|j\rangle \otimes I_2) = \frac{1}{2}I_2. \tag{19}$$

One can verify that, the value of ρ_B will not be affected if $|\psi^+\rangle$ is replaced by any one of $|\phi^+\rangle$, $|\phi^-\rangle$, or $|\psi^-\rangle$. In this sense, even if the intruder obtains particle B, he still cannot decode the message by Alice at the beginning.

In the following, we investigate the security of dense coding protocol, when one of the two particles is obtained by an intruder. It results in the case that Alice and Bob do not merge their qubits. However the intruder still cannot decode the message as well, and it makes the dense coding protocol have a higher security than that of classical counterpart. For instance, assume that the intruder acquires the particle of Alice while it is supposed to be transported to Bob. The conclusion is that, according to the following equation, without the merger of two particles, the intruder still cannot obtain any information about the message. The reason is that by tracing out the system B, the intruder finds that particle A is in the identity state, i.e.,

$$\rho_A = \sum_{j=0}^{1}(I_2 \otimes \langle j|)|\psi^+\rangle\langle\psi^+|(I_2 \otimes |j\rangle) = \frac{1}{2}I_2. \tag{20}$$

One can verify that, the value of ρ_A will not be affected if $|\psi^+\rangle$ is replaced by any one of $|\phi^+\rangle$, $|\phi^-\rangle$, or $|\psi^-\rangle$. Hence, no message on the coding of Alice can be obtained by the intruder. The only way the intruder may figure out the coding of Alice is to obtain both particles from Alice and Bob, and measure them using the Bell basis, just like what Bob does in the final step of dense coding. This is practically more difficult.

3. Result on multiple transfer of classical messags by dense coding

By using sufficient resource between Alice and Bob, one can send $2n$ bits of classical information from Alice to Bob. The cost is n Bell states, $2n$ Hadamard gates and $2n$ CNOT gates, see Figure 4, because each curve between Alice and Bob represents a standard dense coding protocol. In this section, we discuss whether the above-mentioned cost is necessary.

In order to simplify our discussion, we begin still by a protocol between Alice and Bob, though we shall change the quantum channel, i.e., some noisy two-qubit pure states. To transfer multiple classical information via quantum dense coding protocol, we may use another quantum state to replace $|\phi^+\rangle$. In this case, we use

$$|\psi(\alpha)\rangle = \cos\alpha|0,0\rangle + \sin\alpha|1,1\rangle, \tag{21}$$

where $\alpha \in [0, \pi/2]$. To prepare this state, we need to use the equation below.

$$CNOT\left(\begin{bmatrix}\cos\alpha & \sin\alpha\\ \sin\alpha & -\cos\alpha\end{bmatrix} \otimes I_2\right)|0,0\rangle. \tag{22}$$

The entanglement of this state can be evaluated using the well-known formula in terms of von Neumann entropy, i.e.,

$$E\big(\psi(\alpha)\big) = -(\cos\alpha)^2\log_2(\cos\alpha)^2 - (\sin\alpha)^2\log_2(\sin\alpha)^2. \tag{23}$$

One can straightforwardly show that the entanglement reaches the maximum when $\alpha = \pi/4$. In this case, the state $|\psi(\pi/4)\rangle$ becomes exactly the state $|\varphi^+\rangle$ in equation (12). Hence, the current protocol is an altered protocol from the standard dense coding. However, as the entanglement of state $|\psi(\alpha)\rangle$ becomes less in terms of Figure 3, it is natural to conjecture that the fidelity of transferring classical messages from Alice to Bob may be smaller than one. In the following we investigate the details of above conjecture. For convenience, we still employ the four unitary gates $I, \sigma_x, \sigma_z,$ and $i\sigma_y$ in the standard dense coding, while the quantum channel becomes the state $|\psi(\alpha)\rangle$. If we employ I, the result is

$$I \otimes I|\psi(\alpha)\rangle = |\psi(\alpha)\rangle = \cos\alpha|0,0\rangle + \sin\alpha|1,1\rangle. \tag{24}$$

If we employ σ_x, the result is

$$\sigma_x \otimes I|\psi(\alpha)\rangle = |\phi^+(\alpha)\rangle = \cos\alpha|1,0\rangle + \sin\alpha|0,1\rangle. \tag{25}$$

If we employ σ_z, the result is

$$\sigma_z \otimes I|\psi(\alpha)\rangle = |\psi^-(\alpha)\rangle = \cos\alpha|0,0\rangle - \sin\alpha|1,1\rangle. \tag{26}$$

If we employ $i\sigma_y$, the result is

$$i\sigma_y \otimes I|\psi(\alpha)\rangle = |\phi^-(\alpha)\rangle = \cos\alpha|1,0\rangle - \sin\alpha|0,1\rangle. \tag{27}$$

While investigating the security of our altered dense coding protocol, we find that as the value of ρ_a will not be affected if $|\psi(\alpha)\rangle$ is replaced by $|\psi^-(\alpha)\rangle$, or $|\phi^+(\alpha)\rangle$ is replaced by $|\phi^-(\alpha)\rangle$, the intruder cannot decode the message by obtaining particle A. The reason is that by tracing out the system B, the intruder finds that particle A is in the diagonal state

$$\begin{gathered}\rho_A = \sum_{j=0}^{1}(I_2 \otimes \langle j|)|\psi(\alpha)\rangle\langle\psi(\alpha)|(I_2 \otimes |j\rangle) = \begin{bmatrix}\cos^2\alpha & 0\\ 0 & \sin^2\alpha\end{bmatrix},\\ \rho_A = \sum_{j=0}^{1}(I_2 \otimes \langle j|)|\phi^+(\alpha)\rangle\langle\phi^+(\alpha)|(I_2 \otimes |j\rangle) = \begin{bmatrix}\sin^2\alpha & 0\\ 0 & \cos^2\alpha\end{bmatrix}.\end{gathered} \tag{28}$$

Hence if the particle A is obtained by an intruder then the latter cannot decode any message with probability one. Nevertheless, the probability that the intruder may obtain the correct code from Alice increases from 25% of the standard dense coding to 50%. It shows that the polluted state in equation (21) may affect the security of quantum protocols.

Next, just like the third step in standard dense coding protocol, Alice sends her particle to Bob. Due to the analysis over the security in the last paragraph, one can see that it is safe if the particle is obtained by an intruder. Suppose the particle reaches Bob safely. We need investigate whether Bob can decode the message by Alice.

In this step, we need to use a quantum measurement (P_1, P_2). This measurement is physically realizable when the matrices P_1, P_2 satisfy $P_j \geq 0$ and $P_1 + P_2 = I_4$. The expression of P_1 and P_2 are chosen as follows.

$$\begin{gathered}P_1 = |0,0\rangle\langle 0,0| + |1,1\rangle\langle 1,1|,\\ P_2 = |0,1\rangle\langle 0,1| + |1,0\rangle\langle 1,0|.\end{gathered} \tag{29}$$

While applying this measurement, (P_1, P_2) should be applied to equation (24)-(27), and the result will be the multiplication of either P_1 or P_2. The result of all the multiplication is shown below.

$$\begin{gathered}
P_1|\psi(\alpha)\rangle = \cos\alpha|0,0\rangle + \sin\alpha|1,1\rangle \\
P_1|\phi^+(\alpha)\rangle = 0 \\
P_1|\psi^-(\alpha)\rangle = \cos\alpha|0,0\rangle - \sin\alpha|1,1\rangle \\
P_1|\phi^-(\alpha)\rangle = 0 \\
P_2|\psi(\alpha)\rangle = 0 \\
P_2|\phi^+(\alpha)\rangle = \cos\alpha|1,0\rangle + \sin\alpha|0,1\rangle \\
P_2|\psi^-(\alpha)\rangle = 0 \\
P_2|\phi^-(\alpha)\rangle = \cos\alpha|1,0\rangle - \sin\alpha|0,1\rangle
\end{gathered} \tag{30}$$

The result shows that if Bob obtains the measurement P_1 in a lab, then Bob has two particles in the state $|\psi(\alpha)\rangle$ or $|\psi^-(\alpha)\rangle$. Similarly if Bob obtains the measurement P_2 in a lab, then Bob has two particles in the state $|\phi^+(\alpha)\rangle$ or $|\phi^-(\alpha)\rangle$. In this step, if Bob obtains the measurement P_2, by applying operation $\sigma_x \otimes I_2$, he can transform the result to the measurement P_1. So it suffices to study the case when measurement P_1 is obtained by Bob.

On the other hand, one can show that if P_1 is obtained then one still cannot claim whether the target state is $|\psi(\alpha)\rangle$ or $|\psi^-(\alpha)\rangle$ with probability one. In this case, one can only maximize the success probability for identifying the correct states. So we propose the following function.

$$\begin{aligned}
q(Q,\alpha) := & \quad (\cos\alpha\langle 0,0| + \sin\alpha\langle 1,1|)Q(\cos\alpha|0,0\rangle + \sin\alpha|1,1\rangle) \\
+ & \quad (\cos\alpha\langle 0,0| - \sin\alpha\langle 1,1|)(I_4 - Q)(\cos\alpha|0,0\rangle - \sin\alpha|1,1\rangle).
\end{aligned} \tag{31}$$

Here, Q is a measurement element, i.e., a positive semidefinite matrix such that $I_4 - Q \geq 0$. In this equation, when $\alpha = \pi/4$ and $Q = (\cos\alpha|0,0\rangle + \sin\alpha|1,1\rangle)(\cos\alpha\langle 0,0| + \sin\alpha\langle 1,1|)$, we can obtain $q(Q,\alpha) = 2$. However if $\alpha \neq \pi/4$, then we need to find another Q so that $q(Q,\alpha)$ is at the maximum value for the sake of decoding.

Using the information above, we can make the following calculation.

$$\begin{aligned}
q(Q,\alpha) = & \quad I + (\cos\alpha\langle 0,0| + \sin\alpha\langle 1,1|)Q(\cos\alpha|0,0\rangle + \sin\alpha|1,1\rangle) \\
- & \quad (\cos\alpha\langle 0,0| - \sin\alpha\langle 1,1|)Q(\cos\alpha|0,0\rangle - \sin\alpha|1,1\rangle) \\
= & \quad 2\cos\alpha\sin\alpha\langle 0,0|Q|1,1\rangle + 2\cos\alpha\sin\alpha\langle 1,1|Q|0,0\rangle + I \\
\leq & \quad 2\cos\alpha\sin\alpha + 1.
\end{aligned} \tag{32}$$

Hence, the probability is upper bounded by $\sin 2\alpha + 1$, which reaches the maximum namely two only if $\alpha = \pi/4$. It coincides with the known fact that only the maximal entangled state makes the dense coding protocol work with probability one.

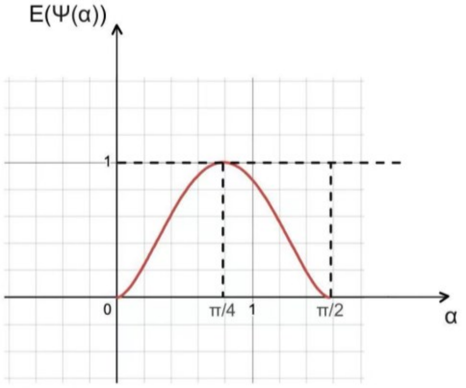

Figure 3: The entanglement of quantum state $|\psi(\alpha)\rangle$.

In this section, we investigate how to transfer more classical messages from Alice to Bob, by using a noisy channel (i.e., non-maximally entangled state of high dimension), see Figure 4. For convenience, we shall only consider the case of two systems. For convenience, we present Eqs. equation (32)-(39). In particular, the 4×4 state in equation (33) means the quantum resource shared by Alice and Bob under two-use of quantum channels. The gate X in equation (34) means the two-qubit X gate, and the gate Z in equation (34) means the two-qubit Z gate. Then the gate $M_{j,k}$ in equation (36) means the physical operation to be carried out by Alice, who need to encode 16 classical messages one-to-one correspond to $M_{j,k}$. This is similar to the standard dense coding protocol, though which involves only one-qubit gate. Hence, the resulting states of Alice and Bob become the 16 states $|\beta_{j,k}\rangle$ in equation (37). Similar to the standard dense coding, Alice need to pass her particle to Bob, who need decode Alice's messages by figuring out the expression of $|\beta_{j,k}\rangle$. This is equivalent to maximize the function in equation (38), under the constraint in equation (39). Note that the matrices Q_j are positive semidefinite matrices, so it is possible to maximize the function by expressing $Q_4 = I_{16} - \sum_{j=2}^{3} Q_j$, just like the case in equation (32). The detailed investigation need a further exploration by using matrix theory assisted by existing optimization programs.

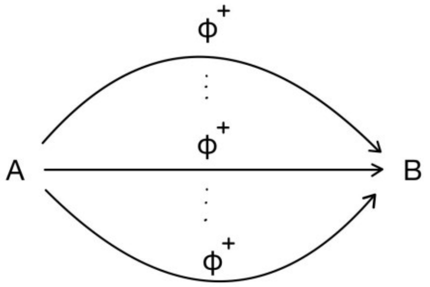

Figure 4: The multiple transfer of classical information via quantum dense coding protocol by using many copies of Bell states between Alice and Bob.

For example, can we use a high-dimensional state and CNOT gate to implement the DC scheme, and try to decompose the state and scheme?

$$|\psi(a_0, a_1, a_2, a_3)\rangle = a_0|0,0\rangle + a_1|1,1\rangle + a_2|2,2\rangle + a_3|3,3\rangle \tag{33}$$

$$X = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{34}$$

$$Z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{bmatrix} \tag{35}$$

$$M_{j,k} = X^j Z^k, \quad j,k = 0,1,2,3 \tag{36}$$

$$\left(M_{j,k} \otimes I_4\right)|\psi(a_0, a_1, a_2, a_3)\rangle = |\beta_{j,k}\rangle \tag{37}$$

$$q(Q_0, Q_1, Q_2, Q_3, a_0, a_1, a_2, a_3) = \sum_{k=0}^{3} \langle \beta_0, k|Q_k|\beta_{0,k}\rangle \tag{38}$$

$$Q_j \geq 0, \sum_{j=0}^{3} Q_j = I_{16} \tag{39}$$

4. Conclusion

We have studied the noisy and multi-use of quantum channels for dense coding protocols. The probability function for success transfer has played a key role in this process. We have mathematically maximized the functions in terms of the quantum measurements taken by Bob. The next step in this direction is to study more complex channels such as mixed entangled states, in terms of various quantum measurements such as von Neumann projections. Another interesting problem is to study whether the classical message transferable by channels can be less impacted by the noise.

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