Proceedings of the 2nd International Conference on Mathematical Physics and Computational Simulation
DOI: 10.54254/2753-8818/39/20240610

Singular value decomposition: A useful technique for image denoising

Kejia Xing

Sun Yat-sen University, No. 135, Xingang West Road, Guangzhou, 510275, China

413812096@qq.com

Abstract. A key function of image processing is picture denoising, which improves the quality of images by eliminating extraneous noise while keeping crucial information in tact. Singular Value Decomposition (SVD) is a linear algebraic technique that reduces the original data's complexity and scale by breaking down the matrices and extracting the important information. With the power of decomposition which utilizes the non-local self-similarity property of an image to achieve satisfactory denoising performance, SVD denoising has become a potent tool in image processing. In this paper, SVD is outlined and its working, applications, and challenges as a denoising technique in image denoising are discussed. The author discovered that Singular Value Decomposition can be a significant factor in image denoising by applying it to the image. As a result, Singular Value Decomposition could be thought as a helpful image denoising approach in the image processing sequence that will raise the image's Peak Signal-to-Noise Ration (PSNR) and improve the quality of the image.

Keywords: Image Processing, Image Denoising, Singular Value Decomposition, Denoising Technique

1. Introduction

In many image processing applications, including image reconstruction, image enhancement, and image compression, image denoising is a crucial step. The fundamental concept behind image denoising is the extraction of undesirable noise from an image using a variety of computational or mathematical techniques, all the while maintaining the image's essential information. A number of things, such as transmission faults, poor lighting, and malfunctioning camera sensors, can result in noise in an image. Furthermore, keeping image detail while removing noise is a difficult challenge. Numerous denoising techniques have been put forth over time, such as statistical techniques, linear filters and more advanced techniques based on deep learning. A matrix is broken down into the product of three matrices in the process on Singular Value Decomposition:a left singular matrix, a diagonal matrix with the singular values, and a right singular matrix [1-3]. The theory behind Singular Values Decomposition is presented in this work along with the example of its application to image denoising. An example of SVD denoising is applied to a image with noise, and the resultant denoising is displayed to demonstrate the effectiveness of SVD denoising in image denoising. This paper finishes with some thoughts on image denoising and SVD denoising. Since noise is a natural part of all photos, learning how to effectively reduce noise in an image is crucial to enhancing its quality and legibility. In addition, picture denoising is a crucial aspect of image processing, and image processing sequences can make extensive use of a good denoising technique.

2. The Principle of SVD in Image Denoising

The SVD denoising [4][5] starts with the process of decomposition. As mentioned above in this paper, the SVD process [6] of an mxn matrix could be expressed as $A = U\Sigma V^T$, and it is important to find these three matrices. In order to get these three matrices, matrix multiplication [7] is performed between the transpose of A and A, In this paper, an nxn square matrix A^TA can be obtained. Since A is a square matrix, in this paper it can be eigen-decomposed [8], and the eigenvalues and eigenvectors obtained satisfy the following equation:

$$(A^TA)_{v_i} = \lambda_i v_i \tag{1}$$

This gives the n eigenvalues of the matrix A^TA and the corresponding n eigenvector of the matrix. Arranging all the eigenvector of the matrix A^TA as an nxn matrix V which yields the V matrix in the SVD formula. It is common to refer to each eigenvector in V as the right singular vector of A. Besides, if this paper perform matrix multiplication between A and the transpose of A, then an mxm square matrix AA^T is obtained. Since AA^T is an mxm square matrix, an eigen-decomposition can be conducted, and the resulting eigenvalues and eigenvectors satisfy the following equation [9]:

$$(AA^T)_{u_i} = \lambda_i u_i \tag{2}$$

In this way, the m eigenvalues of the matrix AA^T and the corresponding m eigenvectors u of the matrix can be obtained. Arranging all the eigenvectors of the matrix as an mxm matrix U yields the matrix U in the SVD formula [10]. In this paper, in particular, each eigenvector in U is considered as the left singular vector of A.

By solving the above two equations, this paper can easily obtain the matrix U and V in the SVD decomposition [11], and the remaining work is to find the diagonal matrix Σ. For the reason that the elements of the matrix Σ are all zero except the diagonal elements are singular values, the finding of matrix Σ can be easily transformed to the computation of each singular value σ. And it is noticed that the SVD formula is $A = U\Sigma V^T$, which can be written as $AV = U\Sigma V^TV$. This is because the matrix V is an nxn orthogonal matrix, and the multiplication between the transpose of V and V is equal to a diagonal matrix with all diagonal elements being one. Therefore, $AV = U\Sigma V^TV$ can be written as $AV = U\Sigma$. And if this paper match all the diagonal elements one-to-one, $Av_i = \sigma_i u_i$ can be obtained, The paper then can compute the value of σ_i by solving Av_i/u_i. In this way, all the singular value of the matrix A can be computed, and all the singular values are arranged on the mxn diagonal as the matrix Σ.

Using this technique, the SVD process can be performed to all the matrix A. Regarding singular values, they are similar to the eigenvalues in eigen-decomposition. In the singular value matrix, they are arranged in descending order. Additionally, the singular values decrease particularly fast. In many cases, the sum of the top 10% or even 1% of singular values accounts for over 99% of the total sum of all singular values. In other words, this paper can also approximate the matrix by using the largest k singular values and their corresponding left and right singular vectors. In short, this process can be written as [12]:

$$A_{m\times n} = U_{m\times m}\Sigma_{m\times n}V_{n\times n}^T \approx U_{m\times k}\Sigma_{k\times k}V_{k\times n}^T \tag{3}$$

Here, k is much smaller than n, which means a large matrix A can be represented by three smaller matrices: $U_{m\times k}, \Sigma_{k\times k}$, and $V_{k\times n}^T$. As shown in the Figure 1 [13], the matrix A can now be approximately described using only the three small matrices in the gray areas:

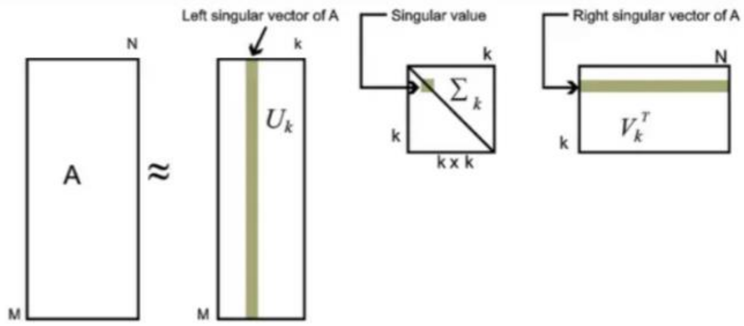

Figure 1. Map of SVD Decomposition

Due to this important property, SVD can be used for PCA dimension reduction, data compression, and denoising.

The SVD-based denoising technique belongs to a category of subspace algorithms. Briefly, this paper aims to decompose the vector space of the noisy signal into two subspace, one dominated by the pure signal the the other by the noise signal. By simply removing the noisy signal vector components that fall within the noisy space, this paper can estimate the pure signal. To achieve this decomposition of the noisy signal vector space into a signal subspace and a noise subspace, it can utilize orthogonal matrix decomposition techniques in linear algebra, particularly Singular Value Decomposition.

In order to perform SVD decomposition as a denoising method, the paper assumes that the image can be stored as a matrix Y with noisy signal in the matrix, then the matrix Y can be written as Y= X + D, in which X represents the matrix containing the pure signal data, and matrix D contains the noisy data. The purpose of denoising is to recover the signal contained in X from the given noisy signal matrix Y. This is equivalent to the following question: Is it possible to recover the signal subspace from Y using SVD.

In order to solve this problem, it is assumed that the matrix X is of rank deficiency, which means the rank of mxn matrix X is smaller than m. Besides, it is assumed that this paper can perform SVD decomposition on matrix X and the process can be written as the equation below [14]:

$$\mathrm{X} = \mathrm{U_x \Sigma_x V_x} = [\mathrm{U_{x1}} \quad \mathrm{U_{x2}}] \begin{bmatrix} \Sigma_{x1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{V_{x1}} \\ \mathrm{V_{x2}} \end{bmatrix} \tag{4}$$

Here, $\mathrm{U_{x1}}$ is an nxr matrix (in this paper it is assumed that the rank of the matrix X is r, which satisfies the condition that it is less than m), $\mathrm{U_{x1}}$ is a rxr matrix, $\mathrm{V_{x1}}$ is a rxm matrix and $\mathrm{V_{x2}}$ is a (n-r)xm matrix due to the fact that the space spanned by the columns of matrix X and matrix Σ_{x1} is typically referred to as the signal subspace, which is also known as the column space of X. Therefore, when this SVD decomposition can be used to rewrite the equation Y= X + D [15]:

$$\mathrm{Y = X + D = X + D(V_{x1}{V_{x1}}^H + V_{x2}{V_{x2}}^H) = (XV_{x1} + DV_{x1}){V_{x1}}^H + DV_{x2}{V_{x2}}^H} \tag{5}$$

Here, the paper have $\mathrm{V_{x1}{V_{x1}}^H + V_{x2}{V_{x2}}^H} = \mathrm{I}$ due to the characteristic of unitary matrix. If it performs SVD decomposition process on $\mathrm{XV_{x1} + DV_{x1}}$ and $\mathrm{DV_{x2}}$, the SVD formula, $\mathrm{XV_{x1} + DV_{x1}}=P_1 S_1 Q_1^H$ and $\mathrm{DV_{x2}}=P_2 S_2 Q_2^H$, can be obtained. Then, the above equation can be written as the following form [16]:

$$\mathrm{Y = X + D} = [P_1 \quad P_2] \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \begin{bmatrix} Q_1^H & {\mathrm{V_{x1}}}^H \\ Q_2^H & {\mathrm{V_{x2}}}^H \end{bmatrix} \tag{6}$$

If $P_1^H P_2$=0, which means the column spaces of P_1 and P_2 are orthogonal, the above equation is a meaningful SVD decomposition of Y. As can be seen from the formula, P_1 is not equal to $\mathrm{U_{x1}}$, so it is not possible to compute the signal subspace directly from the matrix Y.

However, the SVD can be utilized to approximate the signal subspace which is not far from the pure image subspace. If the idea of SVD decomposition is considered, it is known that all the singular values of a matrix A can be written as Σ in such a form that all the diagonal elements are the singular values of the matrix A in descending form and the other elements are all zero. Moreover, the singular values decrease at a particularly rapid rate. In many cases, the sum of the first 10% or even 1% of the singular values accounts for more than 99% of the total sum of all singular values. This means that in this paper it is also possible to use the first r largest singular values to approximately describe the matrix.

In the image denoising process, this paper generally considers the parts with the first r largest singular values to be the original content of the image while the rest of the singular values are always irrelevant content of the image, such as the noise of the image. Therefore, this paper defines a concept called truncated singular value decomposition to process image denoising using SVD technique [17]:

$$A_{m\times n} \approx U_{m\times k}\Sigma_{k\times k}V_{k\times n}^{T} \tag{7}$$

3. The Application of SVD in Image Denoising

To further illustrate this denoising method, Figure 2 can be used as an example to perform SVD decomposition:

Figure 2. Original Image

In order for the SVD decomposition to work, noise was added to the original image above with a Gaussian noise value of 20, yielding Figure 3:

Figure 3. Image with Added Noise

In order to perform SVD decomposition denoising of the image, an attempt is made to transform the image using the first 5% singular values of the image. By this method, Figure 4 with less noise can be obtained:

DOI: 10.54254/2753-8818/39/20240610

Figure 4. Image with First 5% Singular Values

As can be seen in Figure 4, the role of SVD decomposition in the denoising process is to remove some of the noise from the image. If this paper wants to get better SVD denoising performance, more singular values can be used to retain more important information in the original image, and the singular values in the first 10% of the image become better when 5% to 10% is added in this paper (see Figure 5):

Figure 5. Image with First 10% Singular Values

Therefore, the result of SVD decomposition and denoising shows that the SVD technique could be used in image denoising area and have a good performance.

4. Conclusion

In this paper, an efficient image denoising method based on SVD is proposed. Since SVD has the power to decompose the matrix into several parts, one part of the matrix contains the main information of the image while the other part of the matrix contains the noise information of the matrix. The removal of the matrix with noise information plays an important role in image denoising processing. By utilizing the non-local self-similarity of the image and the SVD matrix, the method mentioned in this paper effectively removes the noise while preserving the important image details. The experimental results show that the method in this paper plays an important role in image denoising, making it an important tool for various image processing tasks. In order to improve the performance of SVD-based denoising technique, the combination of deep learning techniques with SVD-based denoising technique can be explored in future work to further improve the denoising performance of SVD. In addition, attention can also be paid to the computational aspects of SVD decomposition because the singular values of matrices are always difficult to compute, and finding a better way to perform SVD decomposition on computers will lead to better performance in SVD denoising.

References

[1] Bhawna Goyal, Ayush Dogra, Sunil Agrawal, B.S. Sohi, Apoorav Sharma, Image denoising review: From classical to state-of-the-art approaches, Information Fusion, Volume 55, 2020
[2] Gu, S., & Timofte, R. (2019). A Brief Review of Image Denoising Algorithms and Beyond. Inpainting and Denoising Challenges.

DOI: 10.54254/2753-8818/39/20240610

[3] Fan, L., Zhang, F., Fan, H., & Zhang, C. (2019). Brief review of image denoising techniques. Visual Computing for Industry, Biomedicine and Art, 2.
[4] Dendrinos M, Bakamidis S, Carayannis G. Speech enhancement from noise: A regenerative approach [J]. Speech Communication, 1991, 10(1): 45-57.
[5] De Moor B. The singular value decomposition and long and short spaces of noisy matrices [J]. IEEE transactions on signal processing, 1993, 41(9): 2826-2838.
[6] He, Yanmin, et al. Adaptive denoising by singular value decomposition. IEEE Signal Processing Letters 18.4 (2011): 215-218.
[7] Zhang, Z., Liu, H. and Guo, Q. Image Restoration Using Probability-Inducing Nuclear Norm Minimization, 2022 IEEE International Conference on Image Processing
[8] S. Zheng, L. Duan, W. Hou and D. Kurlovich, “SVD based Image Actual Resolution Estimation, 2022 5th International Conference on Advanced Electronic Materials, Computers and Software Engineering (AEMCSE), Wuhan, China, 2022
[9] T. Chao, W. Zhihui, W. ZeBin, C. Yufeng and G. Jingping, “Parallel optimization of K-SVD algorithm for image denoising based on Spark, 2016 IEEE 13th International Conference on Signal Processing (ICSP), Chengdu, China, 2016
[10] Z. Zhao, S. Wang, D. Wong, Y. Guo and X. Chen, The sparse and low-rank interpretation of SVD-based denoising for vibration signals, 2020 IEEE International Instrumentation and Measurement Technology Conference (I2MTC), Dubrovnik, Croatia, 2020
[11] D. Wu, Y. Zhao, L. Chen and K. Wang, “Sonar Image Denoising via Adaptive Overcomplete Dictionary Based on K-SVD Algorithm, ” 2013 Sixth International Conference on Business Intelligence and Financial Engineering, Hangzhou, China, 2013
[12] X. Deng and Z. Liu, “Image denoising based on steepest descent OMP and K-SVD, ” 2015 IEEE International Conference on Signal Processing, Communications and Computing (ICSPCC), Ningbo, China, 2015
[13] B. Liu and X. Liao, “Image Denoising and Magnification via Kernel Fitting and Modified SVD, ” 2009 Fifth International Conference on Information Assurance and Security, Xi’an, China, 2009
[14] R. James, A. M. Jolly, C. Anjali and D. Michael, Image Denoising Using Adaptive PCA and SVD, 2015 Fifth International Conference on Advances in Computing and Communications (ICACC), Kochi, India, 2015
[15] J. Orchard, M. Ebrahimi and A. Wong, “Efficient nonlocal-means denoising using the SVD, ” 2008 15th IEEE International Conference on Image Processing, San Diego, CA, USA, 2008
[16] D. Chu, B. Liu, G. Wan, J. Qi and L. Wan, Research on Denoising Method Based on Iterative SVD Decomposition, 2023 China Automation Congress (CAC), Chongqing, China, 2023
[17] M. Seyedebrahim and A. Mansouri, “Non-local means denoising based on SVD basis images, ” 2017 3rd International Conference on Pattern Recognition and Image Analysis (IPRIA), Shahrekord, Iran, 2017